



Research article

Studies on the Rao–Nakra sandwich beam: well-posedness, dynamics, and controllability

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Abstract: In this work, we investigate two distinct control-theoretic problems associated with a linear Rao–Nakra-type sandwich beam. The first problem concerns stabilization under time-varying distributed damping and delayed boundary feedback, while the second addresses boundary controllability under dynamical boundary interactions. Both systems consist of three coupled equations that represent the longitudinal displacements of the outer layers and the transverse displacement of the composite beam, all of which are coupled with boundary conditions. In the first problem, time-varying interior damping and static boundary conditions with time-dependent delays are considered. Then, we establish the existence and uniqueness of solutions for the Cauchy problem associated with the damped system using semigroup theory and a classical result by Kato. Furthermore, employing a Lyapunov-based approach, we prove that the system’s energy decays exponentially, despite the presence of time-varying weights and delays. In the second problem, we consider a boundary linear control system with dynamical boundary conditions, and prove its well-posedness. By deriving an observability inequality for the adjoint system and applying the Hilbert Uniqueness Method (HUM), we show that the system is null controllable. A key contribution of this work lies in handling the full three-equation coupled system, which involves a challenging difficulty due to the delayed or dynamical boundary conditions, resolved via appropriately constructed Lyapunov functionals and intermediate observability inequalities.

Keywords: Rao–Nakra sandwich beam; boundary controllability; exponential stabilization; time-varying delay; Lyapunov method; observability inequality

1. Introduction

1.1. Model formulation and control problems

Various types of beams are widely used in practice. Indeed, we encounter the classical Euler-Bernoulli beam, Rayleigh beam, Timoshenko beam, Bresse beam, sandwich, and micro-beam. One typical example of a sandwich beam is the so-called Rao-Nakra beam. The model describes the dynamics of a three-layer composite beam consisting of two stiff elastic face layers (or outer layers) bonded to a viscoelastic core. This configuration is commonly used in structural applications where vibration suppression and damping are desired, as the viscoelastic layer dissipates energy through shear deformation.

The model was first proposed by Rao and Nakra [1] in 1974 as an extension of classical beam theories, incorporating transverse shear deformation and rotary inertia effects, which are neglected in simpler models such as Euler-Bernoulli or Timoshenko beams. The governing equations couple the longitudinal displacements of the outer layers with the transverse displacement of the beam, linked through the shear strain in the viscoelastic core. This system of partial differential equations captures the interaction between bending, stretching, and shear in sandwich structures. It has become a benchmark model for studying vibration control, stabilization, and active/passive damping mechanisms in smart materials and composite structures.

The purpose of this work is not to analyze a single boundary configuration under multiple objectives, but rather to study two complementary control problems associated with the same Rao-Nakra interior dynamics. Although the two systems considered in this paper share the same coupled PDE structure inside the domain, they correspond to different boundary mechanisms and different control settings. The first system is designed to investigate stabilization effects in the presence of delays and time-dependent damping, whereas the second one is formulated as a boundary control problem involving dynamic boundary interactions. More precisely, we first consider the following system:

$$\left\{ \begin{array}{ll} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) + a_1(t)u_t = 0, & x \in (0, L), t > 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) + a_2(t)v_t = 0, & x \in (0, L), t > 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x + a_3(t)w_t = 0, & x \in (0, L), t > 0, \\ u(0, t) = v(0, t) = w(0, t) = w_x(0, t) = w(L, t) = 0, & t > 0, \\ u_x(L, t) = \alpha_1 u_t(L, t) + \beta_1 u_t(L, t - \tau_1(t)), & t > 0, \\ v_x(L, t) = \alpha_2 v_t(L, t) + \beta_2 v_t(L, t - \tau_2(t)), & t > 0, \\ w_{xx}(L, t) = \alpha_3 w_{tx}(L, t) + \beta_3 w_{tx}(L, t - \tau_3(t)), & t > 0, \\ u_t(L, t - \tau_1(0)) = f_0(t - \tau_1(0)) \in L^2(0, 1), & 0 < t < \tau_1(0), \\ v_t(L, t - \tau_2(0)) = g_0(t - \tau_2(0)) \in L^2(0, 1), & 0 < t < \tau_2(0), \\ w_{tx}(L, t - \tau_3(0)) = h_0(t - \tau_3(0)) \in L^2(0, 1), & 0 < t < \tau_3(0), \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in (0, L), \\ (u_t, v_t, w_t)(x, 0) = (u_1, v_1, w_1)(x), & x \in (0, L), \end{array} \right. \quad (1)$$

where u , v , and w denote the longitudinal displacements of the outer layers and the transverse displacement of the composite. The constants ρ_i, h_i, E_i, I_i ($i = 1, 2, 3$) denote the density, thickness, Young's

modulus, and moments of inertia of each layer, and k is the shear modulus of the middle layer. Moreover, we set

$$\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3, \quad EI = E_1 I_1 + E_3 I_3, \quad \alpha = h_2 + \frac{1}{2}(h_1 + h_3).$$

Furthermore, the boundary conditions involve time-varying delays $\tau_i(t)$, $i = 1, 2, 3$, and we assume the existence of positive constants τ_{0i} , M_i , and $d_i < 1$ such that the delays satisfy the following assumptions:

$$\begin{cases} 0 < \tau_{0i} \leq \tau_i(t) \leq M_i, & \dot{\tau}_i(t) \leq d_i < 1, & \forall t \geq 0, \\ \tau_i \in W^{2,\infty}([0, T]), & & \forall T > 0. \end{cases} \quad (2)$$

The damping coefficients $a_i : \mathbb{R}_+ \rightarrow (0, +\infty)$ are non-increasing C^1 functions and satisfy

$$0 < a_{0i} \leq a_i(t) < b_{0i}, \quad \forall t \geq 0. \quad (3)$$

Finally, the feedback gains α_i, β_i obey

$$\alpha_1 > \frac{|\beta_1|}{2E_1 h_1} \left(\frac{(E_1 h_1)^2 + 1 - d_1}{1 - d_1} \right), \quad 0 \leq d_1 < 1, \quad (4)$$

$$\alpha_2 > \frac{|\beta_2|}{2E_3 h_3} \left(\frac{(E_3 h_3)^2 + 1 - d_2}{1 - d_2} \right), \quad 0 \leq d_2 < 1, \quad (5)$$

$$\alpha_3 > \frac{|\beta_3|}{2EI} \left(\frac{(EI)^2 + 1 - d_3}{1 - d_3} \right), \quad 0 \leq d_3 < 1. \quad (6)$$

Now, consider the energy associated with system (1):

$$\begin{aligned} E(t) = & \frac{1}{2} \left(\rho_1 h_1 \|u_t\|^2 + E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|v_t\|^2 + E_3 h_3 \|v_x\|^2 + \rho h \|w_t\|^2 + EI \|w_{xx}\|^2 \right. \\ & \left. + k \|-u + v + \alpha w_x\|^2 \right) + \frac{|\beta_1|}{2} \tau_1(t) \int_0^1 u_t^2(L, t - \tau(t)\rho) d\rho \\ & + \frac{|\beta_2|}{2} \tau_2(t) \int_0^1 v_t^2(L, t - \tau(t)\rho) d\rho + \frac{|\beta_3|}{2} \tau_3(t) \int_0^1 w_{tx}^2(L, t - \tau(t)\rho) d\rho. \end{aligned} \quad (7)$$

Thereafter, the natural questions appear as follows:

Problem P_1 : Does system (1) admit a solution?

Problem P_2 : If a solution exists for (1), does the time-varying delay feedback ensure exponential decay of the associated energy (7)?

We now consider a boundary controllability problem associated with the same Rao–Nakra interior dynamics, but under a different boundary control configuration. In this setting, the boundary relations are interpreted as dynamic control interactions acting on the coupled beam structure. The resulting system is designed to investigate controllability properties for a strongly coupled Rao–Nakra model under dynamic boundary actuation. We emphasize that the dynamic boundary conditions introduced below are considered from a control-theoretic perspective. Their role is to model boundary interactions and actuator-type effects suitable for the observability and HUM framework adopted in this work, rather than to provide a complete mechanical derivation of the boundary dynamics. In particular,

the boundary acceleration terms appearing in the dynamic boundary relations are interpreted here as abstract boundary control interactions associated with actuator dynamics, rather than as the outcome of a complete mechanical derivation. Thus, the system can be written as

$$\left\{ \begin{array}{ll} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, & x \in (0, L), t > 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, & x \in (0, L), t > 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, & x \in (0, L), t > 0, \\ u(0, t) = v(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0, & t > 0, \\ w_{xx}(L, t) = w_{xxx}(L, t) = 0, & t > 0, \\ u_{tt}(L, t) + u_x(L, t) = f_1(t), & t > 0, \\ v_{tt}(L, t) + v_x(L, t) = f_2(t), & t > 0, \\ w_{tt}(L, t) - u(L, t) + v(L, t) + \alpha w_x(L, t) = f_3(t), & t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in (0, L), \\ (u_t, v_t, w_t)(x, 0) = (u_1, v_1, w_1)(x), & x \in (0, L), \end{array} \right. \quad (8)$$

where $f_i(t)$, $i = 1, 2, 3$, are the boundary controls. It is important to point out that the boundary acceleration traces appearing in (8) should therefore be understood as abstract dynamic boundary variables arising in the boundary control mechanism. Similar dynamic boundary formulations frequently appear in the PDE control literature when modeling actuator dynamics, boundary interactions, or inertial effects concentrated at the boundary.

Although system (8) shares the same interior Rao–Nakra equations as the stabilization model, its boundary structure is substantially different. In particular, the delayed feedback conditions are replaced by dynamic boundary control relations involving boundary acceleration variables. Consequently, the controllability problem studied below should be viewed as a distinct boundary control configuration associated with the same interior coupled dynamics.

Problem P_3 : For the controlled system (8), given $T > 0$ and initial states $(u_0, u_1, v_0, v_1, w_0, w_1)$ in a suitable function space, can one determine control inputs f_1, f_2, f_3 such that the solution (u, v, w) satisfies

$$\begin{aligned} (u, v, w)(x, 0) &= (u_0, v_0, w_0)(x), & x \in (0, L), \\ (u_t, v_t, w_t)(x, 0) &= (u_1, v_1, w_1)(x), & x \in (0, L), \end{aligned}$$

and

$$\begin{aligned} (u, v, w)(x, T) &= (0, 0, 0), & x \in (0, L), \\ (u_t, v_t, w_t)(x, T) &= (0, 0, 0), & x \in (0, L)? \end{aligned}$$

If, for arbitrary $T > 0$, such controls can always drive the system to the equilibrium state, then the system is said to be *null controllable* or *controllable to zero*.

1.2. Background and state of the art

The Rao–Nakra sandwich beam has been extensively studied from analytical and control-theoretic perspectives, especially regarding stabilization and controllability properties. Since the pioneering work in [2], where the Riesz basis property was established for a two-layer beam, several results on

exponential stability and exact controllability have been obtained. Later, a three-layer Rao–Nakra beam was investigated in [3], where exact controllability was proved for sufficiently large control time under suitable parametric assumptions.

More general multilayer configurations were considered in [4,5]. In [4], the authors studied multilayer plates composed of alternating stiff and compliant layers, and established exponential stability through a Riesz basis approach. A related model with interior damping and different boundary conditions was analyzed in [5], where exact controllability was obtained for sufficiently large control time and small damping parameters.

Several works have also addressed stabilization properties under different damping mechanisms and additional physical effects. The influence of magnetic effects in piezoelectric Rao–Nakra systems was investigated in [6,7]. Polynomial stability results were obtained in [8,9], while exponential stability under thermoelastic or delayed damping effects was proved in [10–12]. Non-autonomous dynamics with nonlinear damping and source terms were studied in [13], including the existence of pullback attractors.

Fractional and generalized Rao–Nakra models have likewise attracted considerable attention. Fractional boundary and interior damping mechanisms were considered in [14], yielding polynomial decay estimates depending on the fractional parameters. Generalized Rao–Nakra systems involving several wave equations and one Euler–Bernoulli beam equation were studied in [15]. Stability results involving memory effects, Kelvin–Voigt damping, thermal dissipation, and nonlinear damping mechanisms can also be found in [16–19].

Other works focused on the asymptotic behavior under time-dependent or localized damping effects. In [20], polynomial stability was established when Kelvin–Voigt damping acts only on part of the system, whereas exponential stability with time-varying weights and delays was obtained in [21]. Analytic and polynomial stability results for generalized Rao–Nakra systems were also derived in [22] depending on the damping configuration.

Finally, Rao–Nakra systems posed on the whole line, as well as models involving memory or localized damping effects, have also been investigated. Polynomial decay results on $\mathbb{R}$ were established in [23], while thermoplastic effects and nonlinear damping mechanisms were analyzed in [24]. Exponential, polynomial, and strong stability results with infinite memory terms were obtained in [25]. We also mention the localized damping results in [26,27], where linear and nonlinear frictional damping mechanisms were shown to ensure energy decay. In a different direction, the interaction between nonlinear stabilization mechanisms and nonlinear source terms was studied in [28], where blow-up phenomena, lifespan estimates, and blow-up rates were established.

Despite this extensive literature, several issues remain largely unexplored, particularly for fully coupled Rao–Nakra systems involving simultaneous non-autonomous effects and nonstandard boundary interactions. More precisely, to the best of our knowledge, few results address:

- time-varying distributed damping together with time-dependent boundary delays in a three-component Rao–Nakra system;
- exponential stabilization under delayed boundary feedback and non-autonomous coefficients;
- null controllability for Rao–Nakra systems with dynamic boundary interactions;
- observability inequalities for strongly coupled Rao–Nakra structures under nonstandard boundary dynamics.

The present work addresses these questions using semigroup methods, Lyapunov techniques, observability inequalities, and the Hilbert Uniqueness Method.

1.3. Main results and paper's outline

The first main result of this article shows that the energy $E(t)$ (see (7)) decays exponentially despite the presence of a time-dependent delay. Moreover, an explicit estimate for the decay rate is provided. Of course, we begin by establishing the existence of solutions for a Cauchy problem associated with system (1). This outcome is guaranteed by invoking an abstract theorem due to [29]. With the existence of a solution confirmed, thus providing an answer to Problem P_1 , we can then investigate whether the energy associated with the system decays exponentially. By applying Lyapunov's method, it follows that the energy $E(u(t))$ decays exponentially to zero as $t \rightarrow \infty$, thereby also providing a solution to Problem P_2 .

Before stating our first main result, we shall introduce some notations. In the sequel, we will consider the Lebesgue space $L^2(0, L)$, equipped with its usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Next, we introduce the following Hilbert spaces:

$$H_*^1(0, L) := \{u \in H^1(0, L); u(0) = 0\},$$

$$V_*^2(0, L) := \{w \in H_0^1(0, L) \cap H^2(0, L); w_x(0) = 0\}.$$

Then, we define the space $\mathcal{H}$ as

$$\mathcal{H} = [H_*^1(0, L) \times L^2(0, L)]^2 \times V_*^2(0, L) \times L^2(0, L).$$

For $U = (u, \Psi_1, v, \Psi_2, w, \Psi_3)^\top$, $U^\# = (u^\#, \Psi_1^\#, v^\#, \Psi_2^\#, w^\#, \Psi_3^\#)^\top \in \mathcal{H}$, the space $\mathcal{H}$ is endowed with the inner product

$$\begin{aligned} \langle U, U^\# \rangle_{\mathcal{H}} = & \rho_1 h_1 \langle \Psi_1, \Psi_1^\# \rangle + E_1 h_1 \langle u_x, u_x^\# \rangle + \rho_3 h_3 \langle \Psi_2, \Psi_2^\# \rangle + E_3 h_3 \langle v_x, v_x^\# \rangle \\ & + \rho h \langle \Psi_3, \Psi_3^\# \rangle + EI \langle w_{xx}, w_{xx}^\# \rangle + k \langle -u + v + \alpha w_x, -u^\# + v^\# + \alpha w_x^\# \rangle \end{aligned}$$

and induced norm

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 = & \rho_1 h_1 \|\Psi_1\|^2 + E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|\Psi_2\|^2 + E_3 h_3 \|v_x\|^2 + \rho h \|\Psi_3\|^2 + EI \|w_{xx}\|^2 \\ & + k \|-u + v + \alpha w_x\|^2. \end{aligned}$$

Let the space state

$$\mathcal{H}_1 = \mathcal{H} \times [L^2(0, 1)]^3$$

be equipped with the inner product

$$\langle (U, z_1, z_2, z_3), (U^\#, z_1^\#, z_2^\#, z_3^\#) \rangle_t = \langle U, U^\# \rangle_{\mathcal{H}} + \sum_{i=1}^3 |\beta_i| \tau_i(t) \langle z_i, z_i^\# \rangle_{L^2(0,1)},$$

for any $(U, z_1, z_2, z_3), (U^\#, z_1^\#, z_2^\#, z_3^\#) \in \mathcal{H}_1$.

Now, the first main result can be stated as follows:

Theorem 1.1. *Suppose that the time-dependent delay functions and the time-varying weights satisfy (2) and (3), respectively, and the feedback gains α_i, β_i fulfill (4)–(6), for $i = 1, 2, 3$. Then, there exist three positive constants ζ , λ and μ_* such that the energy $E(t)$ given by (7), along the solution of system (1) in $\mathcal{H}_1$ satisfies*

$$E(t) \leq \zeta e^{-\frac{\lambda}{\mu_*} t} E(0), \quad \text{for all } t \geq 0.$$

On the other hand, the second main result of this paper concerns the boundary controllability of system (8). More precisely, it shows that the system can be driven from any given initial state to the equilibrium state. This provides a positive answer to Problem P_3 .

To define the state space of (8), we introduce the following Hilbert spaces

$$H_*^2(0, L) := \left\{ w \in H^2(0, L); w_x(0) = 0 \right\} \cap \left\{ w \in L^2(0, L); \int_0^L w \, dx = 0 \right\}$$

and

$$H_*^4(0, L) := \left\{ w \in H^4(0, L); w_{xx}(L) = w_{xxx}(L) = w_{xxx}(0) = 0 \right\}.$$

Then, the phase space is defined as

$$\mathcal{H}_2 = \left[H_*^1(0, L) \times L^2(0, L) \right]^2 \times H_*^2(0, L) \times L^2(0, L) \times \mathbb{R}^3.$$

For

$$U = (u, \Psi_1, v, \Psi_2, w, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in \mathcal{H}_2 \quad (9)$$

and

$$U^\# = (u^\#, \Psi_1^\#, v^\#, \Psi_2^\#, w^\#, \Psi_3^\#, \Psi_4^\#, \Psi_5^\#, \Psi_6^\#)^\top \in \mathcal{H}_2, \quad (10)$$

the phase space is endowed with the inner product

$$\begin{aligned} \langle U, U^\# \rangle_{\mathcal{H}_2} = & \rho_1 h_1 \langle \Psi_1, \Psi_1^\# \rangle + E_1 h_1 \langle u_x, u_x^\# \rangle + \rho_3 h_3 \langle \Psi_2, \Psi_2^\# \rangle + E_3 h_3 \langle v_x, v_x^\# \rangle \\ & + \rho h \langle \Psi_3, \Psi_3^\# \rangle + EI \langle w_{xx}, w_{xx}^\# \rangle + k \langle -u + v + \alpha w_x, -u^\# + v^\# + \alpha w_x^\# \rangle \\ & + E_1 h_1 \left(\Psi_4, \Psi_4^\# \right)_\mathbb{R} + E_3 h_3 \left(\Psi_5, \Psi_5^\# \right)_\mathbb{R} + \alpha k \left(\Psi_6, \Psi_6^\# \right)_\mathbb{R} \end{aligned}$$

and induced norm

$$\begin{aligned} \|U\|_{\mathcal{H}_2}^2 = & \rho_1 h_1 \|\Psi_1\|_{L^2}^2 + E_1 h_1 \|u_x\|_{L^2}^2 + \rho_3 h_3 \|\Psi_2\|_{L^2}^2 + E_3 h_3 \|v_x\|_{L^2}^2 + \rho h \|\Psi_3\|_{L^2}^2 + EI \|w_{xx}\|_{L^2}^2 \\ & + k \|-u + v + \alpha w_x\|_{L^2}^2 + E_1 h_1 |\Psi_4|_\mathbb{R}^2 + E_3 h_3 |\Psi_5|_\mathbb{R}^2 + \alpha k |\Psi_6|_\mathbb{R}^2. \end{aligned}$$

Our second main result is stated below:

Theorem 1.2. *Let a time $T > 0$ and given*

$$(u_0, u_1, v_0, v_1, w_0, w_1, u_1(L), v_1(L), w_1(L))^\top \in D(\mathcal{P}),$$

one can always find control inputs $f_i \in L^2(0, T)$, $i = 1, 2, 3$, such that (8) admits a unique solution

$$U = (u, u_t, v, v_t, w, w_t, u_t(L), v_t(L), w_t(L))^\top \in C([0, T]; D(\mathcal{P}))$$

satisfying

$$(u(T), u_t(T), v(T), v_t(T), w(T), w_t(T), u_t(L, T), v_t(L, T), w_t(L, T)) = (0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Here, $\mathcal{P}$ denotes the operator associated with system (8), whose precise definition is given later in (52) and (53).

To establish Theorem 1.2, we employ the observation–control duality framework developed by Dolecki and Russell [30], within the setting introduced by Lions [31] (see also [32]). Specifically, we rely on the Hilbert Uniqueness Method (HUM), which provides a fundamental equivalence between controllability and observability.

A central contribution of our work is the study of a Rao–Nakra sandwich beam model involving three coupled equations. This significantly increases the complexity of the analysis, as the model consists of three beams subject to dynamic boundary conditions. These dynamic boundary interactions present substantial analytical challenges in proving both the exponential decay of the energy and the observability inequalities required in the controllability analysis. As mentioned earlier, these difficulties are overcome by developing an appropriate Lyapunov approach and by deriving two intermediate observability inequalities, which play a crucial role in handling the full system.

Accordingly, the two parts of the paper should be interpreted as complementary control-theoretic investigations associated with the same Rao–Nakra interior dynamics, but involving distinct boundary mechanisms and different mathematical objectives. In particular, the controllability problem studied in the second part is formulated as an abstract boundary control configuration associated with the same coupled Rao–Nakra dynamics considered in the stabilization analysis, rather than as a reformulation of the first boundary model.

We conclude the introduction with an outline of the paper. In Section 2, we establish the well-posedness of system (1) using semigroup theory, specifically applying a theorem due to Kato [29], thereby providing a solution to the first problem posed in the introduction. Section 3 is devoted to proving our first main result, showing that the energy associated with system (1) decays exponentially, thus addressing the second main problem introduced in this work. Sections 4 and 5 focus on the properties of the control system (8). In particular, we first establish the well-posedness of this system via semigroup theory, and then derive an observability inequality for the adjoint system associated with (8), which, together with the Hilbert Uniqueness Method, ensures Theorem 1.2. Finally, the paper ends with a conclusion.

2. Well-posedness result for the delayed system

Let us start by considering the following linear abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}U(t) = A(t)U(t), & t > 0, \\ U(0) = U_0, & t > 0, \end{cases} \quad (11)$$

where $A(t): D(A(t)) \subset H \rightarrow H$ is densely defined. If $D(A(t))$ is independent of time t , i.e., $D(A(t)) = D(A(0))$, for $t > 0$. Then, the next theorem ensures the existence and uniqueness of the Cauchy problem (11).

Theorem 2.1 ([29]). *Assume that:*

1. $\mathcal{Z} = D(A(0))$ is a dense subset of H and $D(A(t)) = D(A(0))$, for all $t > 0$.
2. $A(t)$ generates a strongly continuous semigroup on H . Moreover, the family $\{A(t): t \in [0, T]\}$ is stable with stability constants C , m independent of t .

3. $\partial_t A(t)$ belongs to $L_*^\infty([0, T], B(\mathcal{Z}, H))$, the space of equivalent classes of essentially bounded, strongly measure functions from $[0, T]$ into the set $B(\mathcal{Z}, H)$ of bounded operators from $\mathcal{Z}$ into H .

Then, problem (11) has a unique solution $U \in C([0, T], \mathcal{Z}) \cap C^1([0, T], H)$ for any initial datum in $\mathcal{Z}$.

The task ahead is to apply the above result to ensure the existence of solutions for system (1). Arguing in a classical way, for instance, [33] and [34, 35], let us define the auxiliary variables

$$\begin{cases} z_1(\rho, t) = u_t(L, t - \tau_1(t)\rho), \\ z_2(\rho, t) = v_t(L, t - \tau_2(t)\rho), \\ z_3(\rho, t) = w_{tx}(L, t - \tau_3(t)\rho), \end{cases} \quad (12)$$

which satisfy the following transport equations

$$\begin{cases} \tau_1(t)z_{1,t}(\rho, t) + (1 - \dot{\tau}_1(t)\rho)z_{1,\rho}(\rho, t) = 0, & \rho \in (0, 1), t > 0, \\ z_1(0, t) = u_t(L, t), \quad z_1(\rho, 0) = f_0(-\tau_1(0)\rho), & \rho \in (0, 1), t > 0, \end{cases} \quad (13)$$

$$\begin{cases} \tau_2(t)z_{2,t}(\rho, t) + (1 - \dot{\tau}_2(t)\rho)z_{2,\rho}(\rho, t) = 0, & \rho \in (0, 1), t > 0, \\ z_2(0, t) = v_t(L, t), \quad z_2(\rho, 0) = g_0(-\tau_2(0)\rho), & \rho \in (0, 1), t > 0, \end{cases} \quad (14)$$

$$\begin{cases} \tau_3(t)z_{3,t}(\rho, t) + (1 - \dot{\tau}_3(t)\rho)z_{3,\rho}(\rho, t) = 0, & \rho \in (0, 1), t > 0, \\ z_3(0, t) = w_{tx}(L, t), \quad z_3(\rho, 0) = h_0(-\tau_3(0)\rho), & \rho \in (0, 1), t > 0, \end{cases} \quad (15)$$

respectively.

Now, we introduce the new variables

$$\Psi_1 = u_t, \quad \Psi_2 = v_t, \quad \text{and} \quad \Psi_3 = w_t,$$

and define the vector functions

$$V = (u, \Psi_1, v, \Psi_2, w, \Psi_3, z_1, z_2, z_3)^\top.$$

Therefore, systems (1), (13), (14) and (15) can be reformulated as an abstract Cauchy problem

$$\begin{cases} V_t = \mathcal{A}(t)V \\ V(0) = V_0 = (u_0, u_1, v_0, v_1, w_0, w_1, f_0(-\tau_1(0)\cdot), g_0(-\tau_2(0)\cdot), h_0(-\tau_3(0)\cdot))^\top, \end{cases} \quad (16)$$

where the time-dependent operator

$$\mathcal{A}(t): D(\mathcal{A}(t)) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_1$$

is given by

$$\mathcal{A}(t)V := \begin{pmatrix} \Psi_1 \\ \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - a_1(t)\Psi_1] \\ \Psi_2 \\ \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - a_2(t)\Psi_2] \\ \Psi_3 \\ \frac{1}{\rho h} [-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x - a_3(t)\Psi_3] \\ \frac{\dot{\tau}_1(t)\rho - 1}{\tau_1(t)} z_{1,\rho} \\ \frac{\dot{\tau}_2(t)\rho - 1}{\tau_2(t)} z_{2,\rho} \\ \frac{\dot{\tau}_3(t)\rho - 1}{\tau_3(t)} z_{3,\rho} \end{pmatrix}, \quad (17)$$

with domain defined by

$$D(\mathcal{A}(t)) = \left\{ \begin{array}{l} (u, \Psi_1, v, \Psi_2, w, \Psi_3, z_1, z_2, z_3)^\top \in \mathcal{H}_1; \\ (u, v, w) \in [H^2(0, L)]^2 \times H^4(0, L), \\ (\Psi_1, \Psi_2, \Psi_3) \in [H_*^1(0, L)]^2 \times V_*^2(0, L), (z_1, z_2, z_3) \in H^1(0, L), \\ w(L) = 0, \\ z_1(0) = \Psi_1(L), u_x(L) = \alpha_1 \Psi_1(L) + \beta_1 z_1(1), \\ z_2(0) = \Psi_2(L), v_x(L) = \alpha_2 \Psi_2(L) + \beta_2 z_2(1), \\ z_3(0) = \Psi_{3,x}(L), w_{xx}(L) = \alpha_3 \Psi_{3,x}(L) + \beta_3 z_3(1) \end{array} \right\}. \quad (18)$$

Now, taking the triplet $\{\mathcal{A}, \mathcal{H}_1, \mathcal{Z}\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$, for some $T > 0$ fixed and $\mathcal{Z} = D(\mathcal{A}(0))$, we can state and prove the well-posedness result of (11) related to $\{\mathcal{A}, \mathcal{H}_1, \mathcal{Z}\}$. This gives us the answer for Problem P_1 mentioned at the beginning of the work.

Theorem 2.2. Assume that α_i and β_i are real constants such that (4)–(6) holds, for $i = 1, 2, 3$. Taking $V_0 \in \mathcal{H}_1$, there exists a unique solution $V \in C([0, +\infty), \mathcal{H}_1)$ to (11) whose operator is defined by (17)–(18). Moreover, if $V_0 \in D(\mathcal{A}(0))$, then $V \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H}_1)$.

Proof. The result is established in a standard manner; see, for instance, [35]. First, we observe that $\mathcal{Z} = D(\mathcal{A}(0))$ forms a dense subset of $\mathcal{H}_1$, and moreover, $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$ for all $t > 0$. Hence, condition (1) of Theorem 2.1 is satisfied.

Regarding condition (2) of Theorem 2.1, we note that straightforward integrations by parts, together with the imposed boundary conditions, yield

$$\langle \mathcal{A}(t)V, V \rangle_t - \kappa(t) \langle V, V \rangle_t \leq \frac{1}{2} (\Psi_1(L, t), z_1(1, t)) \Phi_{\alpha_1 \beta_1}^{d_1} (\Psi_1(L, t), z_1(1, t))^\top$$

$$\begin{aligned}
& + \frac{1}{2} (\Psi_2(L, t), z_2(1, t)) \Phi_{\alpha_2, \beta_2}^{d_2} (\Psi_2(L, t), z_2(1, t))^{\top} \\
& + \frac{1}{2} (\Psi_{3,x}(L, t), z_3(1, t)) \Phi_{\alpha_3, \beta_3}^{d_3} (\Psi_{3,x}(L, t), z_3(1, t))^{\top},
\end{aligned}$$

where

$$\kappa(t) = \sum_{i=1}^3 \frac{(\dot{\tau}_i(t)^2 + 1)^{1/2}}{2\tau_i(t)}, \quad (19)$$

$$\Phi_{\alpha_1, \beta_1}^{d_1} = \begin{pmatrix} -2E_1 h_1 \alpha_1 + |\beta_1| & -E_1 h_1 \beta_1 \\ -E_1 h_1 \beta_1 & |\beta_1|(d_1 - 1) \end{pmatrix}, \quad (20)$$

$$\Phi_{\alpha_2, \beta_2}^{d_2} = \begin{pmatrix} -2E_3 h_3 \alpha_2 + |\beta_2| & -E_3 h_3 \beta_2 \\ -E_3 h_3 \beta_2 & |\beta_2|(d_2 - 1) \end{pmatrix}, \quad (21)$$

and

$$\Phi_{\alpha_3, \beta_3}^{d_3} = \begin{pmatrix} -2EI\alpha_3 + |\beta_3| & -EI\beta_3 \\ -EI\beta_3 & |\beta_3|(d_3 - 1) \end{pmatrix}. \quad (22)$$

Owing to (4), (5), and (6), it follows that the matrices $\Phi_{\alpha_1, \beta_1}^{d_1}$, $\Phi_{\alpha_2, \beta_2}^{d_2}$, and $\Phi_{\alpha_3, \beta_3}^{d_3}$ are negative definite. Consequently, we have

$$\langle \mathcal{A}(t)V, V \rangle_t - \kappa(t) \langle V, V \rangle_t \leq 0,$$

which implies that the operator $\tilde{\mathcal{A}}(t) := \mathcal{A}(t) - \kappa(t)I$ is dissipative.

We now state the following claim:

Claim 2.1. *For all $t \in [0, T]$, the operator $\mathcal{A}(t)$ is maximal. Equivalently, there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}(t)$ is surjective.*

In fact, let us fix $t \in [0, T]$. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, h_1, h_2, h_3)^{\top} \in \mathcal{H}_1$, we seek a solution $V = (u, \Psi_1, v, \Psi_2, w, \Psi_3, z_1, z_2, z_3)^{\top} \in D(\mathcal{A}(t))$ of the equation $(\lambda I - \mathcal{A}(t))V = F$, that is,

$$\left\{ \begin{aligned}
& \lambda u - \Psi_1 = f_1, \\
& \lambda \Psi_1 - \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - a_1(t)\Psi_1] = f_2, \\
& \lambda v - \Psi_2 = f_3, \\
& \lambda \Psi_2 - \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - a_2(t)\Psi_2] = f_4, \\
& \lambda w - \Psi_3 = f_5, \\
& \lambda \Psi_3 - \frac{1}{\rho h} [-EIw_{4x} + \alpha k(-u + v + \alpha w_x)_x - a_3(t)\Psi_3] = f_6, \\
& \lambda z_i + \left(\frac{1 - \dot{\tau}_i(t)\rho}{\tau_i(t)} \right) z_{i,\varphi} = h_i, \quad i = 1, 2, 3, \\
& u(0) = v(0) = w(0) = w_x(0) = w(L) = 0, \\
& z_1(0) = \Psi_1(L), \quad u_x(L) = \alpha_1 \Psi_1(L) + \beta_1 z_1(1), \\
& z_2(0) = \Psi_2(L), \quad v_x(L) = \alpha_2 \Psi_2(L) + \beta_2 z_2(1), \\
& z_3(0) = \Psi_{3,x}(L), \quad w_{xx}(L) = \alpha_3 \Psi_{3,x}(L) + \beta_3 z_3(1).
\end{aligned} \right. \quad (23)$$

One can readily verify that, for any $i = 1, 2, 3$, z_i is given by

$$z_i(\rho) = \begin{cases} z_i(0)e^{-\lambda\tau_i(t)\rho} + \tau_i(t)e^{-\lambda\tau_i(t)\rho} \int_0^\rho e^{\lambda\tau_i(t)\sigma} h_i(\sigma) d\sigma, & \text{if } \dot{\tau}_i(t) = 0, \\ e^{\lambda \frac{\tau_i(t)}{\dot{\tau}_i(t)} \ln(1-\dot{\tau}_i(t)\rho)} \left[z_i(0) + \int_0^\rho \frac{h_i(\sigma)\tau_i(t)}{1-\dot{\tau}_i(t)\sigma} e^{-\lambda \frac{\tau_i(t)}{\dot{\tau}_i(t)} \ln(1-\dot{\tau}_i(t)\sigma)} d\sigma \right], & \text{if } \dot{\tau}_i(t) \neq 0. \end{cases}$$

Thereby, $z_i(1) = z_i(0)g_{i,0}(t) + g_{h_i}(t)$, in which

$$g_{i,0}(t) = \begin{cases} e^{-\lambda\tau_i(t)}, & \text{if } \dot{\tau}_i(t) = 0, \\ e^{\lambda \frac{\tau_i(t)}{\dot{\tau}_i(t)} \ln(1-\dot{\tau}_i(t))}, & \text{if } \dot{\tau}_i(t) \neq 0, \end{cases}$$

and

$$g_{h_i}(t) = \begin{cases} \tau_i(t)e^{-\lambda\tau_i(t)} \int_0^1 e^{\lambda\tau_i(t)\sigma} h_i(\sigma) d\sigma, & \text{if } \dot{\tau}_i(t) = 0, \\ e^{\lambda \frac{\tau_i(t)}{\dot{\tau}_i(t)} \ln(1-\dot{\tau}_i(t))} \int_0^1 \frac{h_i(\sigma)\tau_i(t)}{1-\dot{\tau}_i(t)\sigma} e^{-\lambda \frac{\tau_i(t)}{\dot{\tau}_i(t)} \ln(1-\dot{\tau}_i(t)\sigma)} d\sigma, & \text{if } \dot{\tau}_i(t) \neq 0. \end{cases}$$

Combining the latter with (23), it follows that $U = (u, \Psi_1, v, \Psi_2, w, \Psi_3)^\top$ is a solution of the system

$$\begin{cases} \lambda u - \Psi_1 = f_1, \\ \lambda \Psi_1 - \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - a_1(t)\Psi_1] = f_2, \\ \lambda v - \Psi_2 = f_3, \\ \lambda \Psi_2 - \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - a_2(t)\Psi_2] = f_4, \\ \lambda w - \Psi_3 = f_5, \\ \lambda \Psi_3 - \frac{1}{\rho h} [-EI w_{4x} + \alpha k(-u + v + \alpha w_x)_x - a_3(t)\Psi_3] = f_6, \end{cases} \quad (24)$$

and satisfies the boundary conditions

$$\begin{cases} u(0) = v(0) = w(0) = w_x(0) = w(L) = 0, \\ u_x(L) = (\alpha_1 + \beta_1 g_{1,0}(t))\Psi_1(L) + \beta_1 g_{h_1}(t), \\ v_x(L) = (\alpha_2 + \beta_2 g_{2,0}(t))\Psi_2(L) + \beta_2 g_{h_2}(t), \\ w_{xx}(L) = (\alpha_3 + \beta_3 g_{3,0}(t))\Psi_{3,x}(L) + \beta_3 g_{h_3}(t). \end{cases}$$

Now, let $\phi_i \in C^\infty([0, L])$, for $i = 1, 2, 3$, be functions such that $\phi_1(0) = \phi_{1,x}(L) = 0$, $\phi_2(0) = \phi_{2,x}(L) = 0$, and $\phi_3(0) = \phi_{3,x}(0) = \phi_3(L) = \phi_{3,xx}(L) = \phi_{3,xxx}(L) = 0$. Next, for $i = 1, 2, 3$, we define functions $\varphi_i(x, \cdot) = \phi_i(x)\beta_i g_{h_i}(\cdot) \in C^\infty([0, L])$ and let

$$\begin{cases} \hat{u} := u - \varphi_1, \\ \hat{v} := v - \varphi_2, \\ \hat{w} := w - \varphi_3. \end{cases}$$

This, together with (24), implies that $\hat{U} = (\hat{u}, \Psi_1, \hat{v}, \Psi_2, \hat{w}, \Psi_3)^\top$ satisfies

$$\begin{cases} \lambda \hat{u} - \Psi_1 = \tilde{f}_1, \\ \lambda \Psi_1 - \frac{1}{\rho_1 h_1} [E_1 h_1 \hat{u}_{xx} + k(-\hat{u} + \hat{v} + \alpha \hat{w}_x) - a_1(t) \Psi_1] = \tilde{f}_2, \\ \lambda \hat{v} - \Psi_2 = \tilde{f}_3, \\ \lambda \Psi_2 - \frac{1}{\rho_3 h_3} [E_3 h_3 \hat{v}_{xx} - k(-\hat{u} + \hat{v} + \alpha \hat{w}_x) - a_2(t) \Psi_2] = \tilde{f}_4, \\ \lambda \hat{w} - \Psi_3 = \tilde{f}_5, \\ \lambda \Psi_3 - \frac{1}{\rho h} [-EI \hat{w}_{4x} + \alpha k(-\hat{u} + \hat{v} + \alpha \hat{w}_x)_x - a_3(t) \Psi_3] = \tilde{f}_6, \end{cases}$$

where

$$\begin{cases} \tilde{f}_1 = f_1 - \lambda \varphi_1, \\ \tilde{f}_2 = f_2 - \frac{1}{\rho_1 h_1} [-E_1 h_1 \varphi_{1,xx} + k(\varphi_1 - \varphi_2 - \alpha \varphi_{3,x})], \\ \tilde{f}_3 = f_3 - \lambda \varphi_2, \\ \tilde{f}_4 = f_4 - \frac{1}{\rho_3 h_3} [-E_3 h_3 \varphi_{2,xx} - k(\varphi_1 - \varphi_2 - \alpha \varphi_{3,x})], \\ \tilde{f}_5 = f_5 - \lambda \varphi_3, \\ \tilde{f}_6 = f_6 - \frac{1}{\rho h} [EI \varphi_{3,xxxx} + \alpha k(\varphi_1 - \varphi_2 - \alpha \varphi_{3,x})_x], \end{cases}$$

as well as the boundary conditions

$$\begin{cases} \hat{u}(0) = \hat{v}(0) = \hat{w}(0) = \hat{w}_x(0) = \hat{w}(L) = 0, \\ \hat{u}_x(L) = (\alpha_1 + \beta_1 g_{1,0}(t)) \Psi_1(L), \\ \hat{v}_x(L) = (\alpha_2 + \beta_2 g_{2,0}(t)) \Psi_2(L), \\ \hat{w}_{xx}(L) = (\alpha_3 + \beta_3 g_{3,0}(t)) \Psi_{3,x}(L). \end{cases}$$

For simplicity, we continue to denote the translated variables by u , v , and w . It follows that $0 < g_{i,0}(t) < 1$ for all $i = 1, 2, 3$ (see, e.g., [36, 37]). Hence, from (4)–(6), we deduce that

$$\tilde{\alpha}_i := \alpha_i + \beta_i g_{i,0}(t) > 0, \quad \text{for each } i = 1, 2, 3.$$

Consequently, establishing Claim 2.1 reduces to proving that $\lambda I - \hat{\mathcal{A}}$ is surjective, where $\hat{\mathcal{A}}$ is defined by

$$\hat{\mathcal{A}}(t)U := \begin{pmatrix} \Psi_1 \\ \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - a_1(t) \Psi_1] \\ \Psi_2 \\ \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - a_2(t) \Psi_2] \\ \Psi_3 \\ \frac{1}{\rho h} [-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x - a_3(t) \Psi_3] \end{pmatrix},$$

with a dense domain

$$D(\hat{\mathcal{A}}(t)) = \left\{ \begin{array}{l} (u, \Psi_1, v, \Psi_2, w, \Psi_3)^\top \in \mathcal{H}; (u, v, w) \in [H^2(0, L)]^2 \times H^4(0, L), \\ (\Psi_1, \Psi_2, \Psi_3) \in [H_*^1(0, L)]^2 \times V_*^2(0, L), w(L) = 0, \\ u_x(L) = \tilde{\alpha}_1 \Psi_1(L), v_x(L) = \tilde{\alpha}_2 \Psi_2(L), w_{xx}(L) = \tilde{\alpha}_3 \Psi_{3,x}(L) \end{array} \right\} \subset \mathcal{H}.$$

Now, observe that the adjoint of $\hat{\mathcal{A}}$, denoted by $\hat{\mathcal{A}}^*$, is given by

$$\hat{\mathcal{A}}^*(t)U := \begin{pmatrix} -\Psi_1 \\ -\frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) - a_1(t)\Psi_1] \\ -\Psi_2 \\ -\frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) - a_2(t)\Psi_2] \\ -\Psi_3 \\ -\frac{1}{\rho h} [-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x - a_3(t)\Psi_3] \end{pmatrix},$$

with

$$D(\hat{\mathcal{A}}^*(t)) = \left\{ \begin{array}{l} (u, \Psi_1, v, \Psi_2, w, \Psi_3)^\top \in \mathcal{H}; (u, v, w) \in [H^2(0, L)]^2 \times H^4(0, L), \\ (\Psi_1, \Psi_2, \Psi_3) \in [H_*^1(0, L)]^2 \times V_*^2(0, L), w(L) = 0, \\ u_x(L) = -\tilde{\alpha}_1 \Psi_1(L), v_x(L) = -\tilde{\alpha}_2 \Psi_2(L), w_{xx}(L) = -\tilde{\alpha}_3 \Psi_{3,x}(L) \end{array} \right\}.$$

One can readily verify that

$$\begin{aligned} \langle \hat{\mathcal{A}}U, U \rangle_{\mathcal{H}} &= -a_1(t)\|\Psi_1\|^2 - a_2(t)\|\Psi_2\|^2 - a_3(t)\|\Psi_3\|^2 \\ &\quad - E_1 h_1 \tilde{\alpha}_1 \Psi_1^2(L) - E_3 h_3 \tilde{\alpha}_2 \Psi_2^2(L) - EI \tilde{\alpha}_3 \Psi_{3,x}^2(L), \end{aligned}$$

and

$$\begin{aligned} \langle \hat{\mathcal{A}}^*U, U \rangle_{\mathcal{H}} &= -a_1(t)\|\Psi_1\|^2 - a_2(t)\|\Psi_2\|^2 - a_3(t)\|\Psi_3\|^2 \\ &\quad - E_1 h_1 \tilde{\alpha}_1 \Psi_1^2(L) - E_3 h_3 \tilde{\alpha}_2 \Psi_2^2(L) - EI \tilde{\alpha}_3 \Psi_{3,x}^2(L). \end{aligned}$$

This guarantees that the operators $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^*$ are dissipative and hence, the desired result follows from the Lumer-Phillips Theorem (see, e.g., [38]), which establishes Claim 2.1.

Consequently, $\tilde{\mathcal{A}}(t)$ generates a strongly continuous semigroup on $\mathcal{H}$, and the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ forms a stable family of generators in $\mathcal{H}$ with a stability constant independent of t . Therefore, condition (2) of Theorem 2.1 is satisfied.

Lastly, since $\tau_i \in W^{2,\infty}([0, T])$ for all $T > 0$, and $i = 1, 2, 3$, we reach that

$$\kappa(t) = \sum_{i=1}^3 \left[\frac{\ddot{\tau}_i(t) \dot{\tau}_i(t)}{2\tau_i(t) (\dot{\tau}_i(t)^2 + 1)^{1/2}} - \frac{\dot{\tau}_i(t) (\dot{\tau}_i(t)^2 + 1)^{1/2}}{2\tau_i(t)^2} \right]$$

is bounded on $[0, T]$, for all $T > 0$, and

$$\frac{d}{dt} \mathcal{A}(t)V = \begin{pmatrix} 0 \\ -\dot{a}_1(t)\Psi_1 \\ 0 \\ -\dot{a}_2(t)\Psi_2 \\ 0 \\ -\dot{a}_3(t)\Psi_3 \\ \frac{\ddot{\tau}_1(t)\tau_1(t)\rho - \dot{\tau}_1(t)(\dot{\tau}_1(t)\rho - 1)}{\tau_1(t)^2} z_{1,\rho} \\ \frac{\ddot{\tau}_2(t)\tau_2(t)\rho - \dot{\tau}_2(t)(\dot{\tau}_2(t)\rho - 1)}{\tau_2(t)^2} z_{2,\rho} \\ \frac{\ddot{\tau}_3(t)\tau_3(t)\rho - \dot{\tau}_3(t)(\dot{\tau}_3(t)\rho - 1)}{\tau_3(t)^2} z_{3,\rho} \end{pmatrix}.$$

Moreover, the coefficients of $z_{i,\rho}$ and Ψ_i are bounded on $[0, T]$ for $i = 1, 2, 3$, so that the regularity condition (3) of Theorem 2.1 is satisfied.

In summary, we have verified all the assumptions of Theorem 2.1. Hence, for each $V_0 \in D(\mathcal{A}(0))$, the Cauchy problem

$$\begin{cases} \tilde{V}_i(t) = \tilde{\mathcal{A}}(t)\tilde{V}(t), \\ \tilde{V}(0) = V_0, \quad t > 0, \end{cases}$$

admits a unique solution

$$\tilde{V} \in C([0, \infty), \mathcal{H}_1) \cap C([0, \infty), D(\mathcal{A}(0))) \cap C^1([0, \infty), \mathcal{H}_1).$$

Therefore, the solution of (11) is explicitly given by

$$V(t) = e^{\int_0^t \kappa(s) ds} \tilde{V}(t),$$

which proves Theorem 2.2. $\square$

We also observe that the energy $E(t)$ associated with system (1) satisfies the following relation.

Proposition 2.1. Suppose α_i and β_i are real constants such that (4)–(6) holds for $i = 1, 2, 3$. Then, for any mild solution of (16), the energy $E(t)$ defined by (7) is non-increasing and

$$\begin{aligned} \frac{d}{dt} E(t) = & -a_1(t)\|u_t\|^2 - a_2(t)\|v_t\|^2 - a_3(t)\|w_t\|^2 \\ & + \frac{1}{2} \begin{pmatrix} u_t(L, t) \\ u_t(L, t - \tau_1(t)) \end{pmatrix}^\top \Phi_{\alpha_1, \beta_1}^{\dot{\tau}_1(t)} \begin{pmatrix} u_t(L, t) \\ u_t(L, t - \tau_1(t)) \end{pmatrix} \\ & + \frac{1}{2} \begin{pmatrix} v_t(L, t) \\ v_t(L, t - \tau_2(t)) \end{pmatrix}^\top \Phi_{\alpha_2, \beta_2}^{\dot{\tau}_2(t)} \begin{pmatrix} v_t(L, t) \\ v_t(L, t - \tau_2(t)) \end{pmatrix} \end{aligned} \quad (25)$$

$$+ \frac{1}{2} \begin{pmatrix} w_{tx}(L, t) \\ w_{tx}(L, t - \tau_3(t)) \end{pmatrix}^\top \Phi_{\alpha_3 \beta_3}^{\dot{\tau}_3(t)} \begin{pmatrix} w_{tx}(L, t) \\ w_{tx}(L, t - \tau_3(t)) \end{pmatrix},$$

where the matrices $\Phi_{\alpha_i \beta_i}^{\dot{\tau}_i(t)}$ are given by (20), (21) and (22), respectively, with $\dot{\tau}_i(t)$ instead of d_i , for $i = 1, 2, 3$.

Proof. Considering the energy identity (7), we define

$$E_1(t) = \frac{1}{2} \left(\rho_1 h_1 \|u_t\|^2 + E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|v_t\|^2 + E_3 h_3 \|v_x\|^2 + \rho h \|w_t\|^2 + EI \|w_{xx}\|^2 + k \|-u + v + \alpha w_x\|^2 \right).$$

We multiply equations $(1)_1$, $(1)_2$, $(1)_3$ by u_t , v_t , w_t , respectively. Then, by integrating by parts on $(0, L)$ and using the boundary conditions, we obtain

$$\begin{aligned} \frac{d}{dt} E_1(t) = & -a_1(t) \|u_t\|^2 - a_2(t) \|v_t\|^2 - a_3(t) \|w_t\|^2 \\ & - E_1 h_1 \alpha_1 u_t^2(L, t) - E_1 h_1 \beta_1 u_t(L, t) u_t(L, t - \tau_1(t)) \\ & - E_3 h_3 \alpha_2 v_t^2(L, t) - E_3 h_3 \beta_2 v_t(L, t) v_t(L, t - \tau_2(t)) \\ & - EI \alpha_3 w_{tx}^2(L, t) - EI \beta_3 w_{tx}(L, t) w_{tx}(L, t - \tau_3(t)). \end{aligned} \quad (26)$$

On the other hand, from (12)–(15), for $i = 1, 2, 3$, we deduce that

$$\frac{d}{dt} \left(\frac{|\beta_i|}{2} \tau_i(t) \int_0^1 z_i^2(L, t - \tau_i(t) \rho) d\rho \right) = \frac{|\beta_i|}{2} (\dot{\tau}_i(t) - 1) z_i^2(1, t) + \frac{|\beta_i|}{2} z_i^2(0, t). \quad (27)$$

Thus, from (7), (26) and (27), and using the auxiliary variables (12), we have

$$\begin{aligned} \frac{d}{dt} E(t) = & -a_1(t) \|u_t\|^2 - a_2(t) \|v_t\|^2 - a_3(t) \|w_t\|^2 \\ & \frac{1}{2} \left((-2E_1 h_1 \alpha_1 + |\beta_1|) u_t^2(L, t) - 2E_1 h_1 \beta_1 u_t(L, t) u_t(L, t - \tau_1(t)) \right. \\ & \quad \left. + |\beta_1| (\dot{\tau}_1(t) - 1) u_t^2(L, t - \tau_1(t)) \right) \\ & \frac{1}{2} \left((-2E_3 h_3 \alpha_2 + |\beta_2|) v_t^2(L, t) - 2E_3 h_3 \beta_2 v_t(L, t) v_t(L, t - \tau_2(t)) \right. \\ & \quad \left. + |\beta_2| (\dot{\tau}_2(t) - 1) v_t^2(L, t - \tau_2(t)) \right) \\ & \frac{1}{2} \left((-2EI \alpha_3 + |\beta_3|) w_{tx}^2(L, t) - 2EI \beta_3 w_{tx}(L, t) w_{tx}(L, t - \tau_3(t)) \right. \\ & \quad \left. + |\beta_3| (\dot{\tau}_3(t) - 1) w_{tx}^2(L, t - \tau_3(t)) \right) \\ = & -a_1(t) \|u_t\|^2 - a_2(t) \|v_t\|^2 - a_3(t) \|w_t\|^2 \\ & + \frac{1}{2} \begin{pmatrix} u_t(L, t) \\ u_t(L, t - \tau_1(t)) \end{pmatrix}^\top \Phi_{\alpha_1 \beta_1}^{\dot{\tau}_1(t)} \begin{pmatrix} u_t(L, t) \\ u_t(L, t - \tau_1(t)) \end{pmatrix} \\ & + \frac{1}{2} \begin{pmatrix} v_t(L, t) \\ v_t(L, t - \tau_2(t)) \end{pmatrix}^\top \Phi_{\alpha_2 \beta_2}^{\dot{\tau}_2(t)} \begin{pmatrix} v_t(L, t) \\ v_t(L, t - \tau_2(t)) \end{pmatrix} \\ & + \frac{1}{2} \begin{pmatrix} w_{tx}(L, t) \\ w_{tx}(L, t - \tau_3(t)) \end{pmatrix}^\top \Phi_{\alpha_3 \beta_3}^{\dot{\tau}_3(t)} \begin{pmatrix} w_{tx}(L, t) \\ w_{tx}(L, t - \tau_3(t)) \end{pmatrix}, \end{aligned}$$

obtaining (25). □

Now, we are ready to state the following result:

Proposition 2.2. Assume that α_i and β_i are real constants such that (4)–(6) holds for $i = 1, 2, 3$. Then, for $V = (U, z_1, z_2, z_3)^\top = (u, u_t, v, v_t, w, w_t, z_1, z_2, z_3)^\top$, the solution of (16), the following estimate holds:

$$\begin{aligned} \|U(t)\|_{\mathcal{H}}^2 + \sum_{i=1}^3 |\beta_i| \tau_{0i} \|z_i(\cdot, t)\|_{L^2(0,1)}^2 &\leq \|U_0\|_{\mathcal{H}}^2 + |\beta_1| \tau_1(0) \|f_0(-\tau_1(0)\cdot)\|_{L^2(0,1)}^2 \\ &+ |\beta_2| \tau_2(0) \|g_0(-\tau_2(0)\cdot)\|_{L^2(0,1)}^2 + |\beta_3| \tau_3(0) \|h_0(-\tau_3(0)\cdot)\|_{L^2(0,1)}^2. \end{aligned} \quad (28)$$

Furthermore, for every initial condition $V_0 \in \mathcal{H}_1$, we have that

$$\begin{aligned} &\|u_t(L, \cdot)\|_{L^2(0,T)}^2 + \|v_t(L, \cdot)\|_{L^2(0,T)}^2 + \|w_{tx}(L, \cdot)\|_{L^2(0,T)}^2 + \sum_{i=1}^3 \|z_i(1, \cdot)\|_{L^2(0,T)}^2 \\ &\leq \|U_0\|_{\mathcal{H}}^2 + |\beta_1| \tau_1(0) \|f_0(-\tau_1(0)\cdot)\|_{L^2(0,1)}^2 + |\beta_2| \tau_2(0) \|g_0(-\tau_2(0)\cdot)\|_{L^2(0,1)}^2 \\ &+ |\beta_3| \tau_3(0) \|h_0(-\tau_3(0)\cdot)\|_{L^2(0,1)}^2. \end{aligned} \quad (29)$$

On the other hand, for the initial datum, we have the following estimates

$$\begin{aligned} \|U_0\|_{\mathcal{H}}^2 &\leq \frac{1}{T} \|U\|_{L^2(0,T;\mathcal{H})}^2 + 2 \sum_{i=1}^3 \|a_i\|_{L^\infty(0,T)} \|(u_t, v_t, w_t)\|_{[L^2(0,T;L^2(0,L))]^3}^2 \\ &+ E_1 h_1 (2\alpha_1 + |\beta_1|) \|u_t(L, \cdot)\|_{L^2(0,T)}^2 + E_3 h_3 (2\alpha_2 + |\beta_2|) \|v_t(L, \cdot)\|_{L^2(0,T)}^2 \\ &+ EI (2\alpha_3 + |\beta_3|) \|w_{tx}(L, \cdot)\|_{L^2(0,T)}^2 + \sum_{i=1}^3 |\beta_i| \|z_i(1, \cdot)\|_{L^2(0,T)}^2 \end{aligned} \quad (30)$$

and

$$\begin{cases} \|f_0(-\tau_1(0)\cdot)\|_{L^2(0,1)}^2 \leq C_1(d_1, M_1) (\|z_1(\cdot, T)\|_{L^2(0,1)}^2 + \|z_1(1, \cdot)\|_{L^2(0,T)}^2), \\ \|g_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2 \leq C_2(d_2, M_2) (\|z_2(\cdot, T)\|_{L^2(0,1)}^2 + \|z_2(1, \cdot)\|_{L^2(0,T)}^2), \\ \|h_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2 \leq C_3(d_3, M_3) (\|z_3(\cdot, T)\|_{L^2(0,1)}^2 + \|z_3(1, \cdot)\|_{L^2(0,T)}^2). \end{cases} \quad (31)$$

Proof. Using (25) and the fact that $\Phi_{\alpha_i \beta_i}^{\tau_i(t)}$ are symmetric negative definite matrices, we deduce the existence of a positive constant C , such that

$$\begin{aligned} E'(t) + a_1(t) \|u_t\|^2 + a_2(t) \|v_t\|^2 + a_3(t) \|w_t\|^2 &= \frac{1}{2} \begin{pmatrix} u_t(L, t) \\ z_1(1, t) \end{pmatrix}^\top \Phi_{\alpha_1 \beta_1}^{\tau_1(t)} \begin{pmatrix} u_t(L, t) \\ z_1(1, t) \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} v_t(L, t) \\ z_2(1, t) \end{pmatrix}^\top \Phi_{\alpha_2 \beta_2}^{\tau_2(t)} \begin{pmatrix} v_t(L, t) \\ z_2(1, t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} w_{tx}(L, t) \\ z_3(1, t) \end{pmatrix}^\top \Phi_{\alpha_3 \beta_3}^{\tau_3(t)} \begin{pmatrix} w_{tx}(L, t) \\ z_3(1, t) \end{pmatrix} \\ &\leq -C \left(u_t^2(L, t) + v_t^2(L, t) + w_{tx}^2(L, t) + \sum_{i=1}^3 z_i^2(1, t) \right). \end{aligned}$$

Thus, it follows from the above estimate that

$$E'(t) + u_t^2(L, t) + v_t^2(L, t) + w_{tx}^2(L, t) + \sum_{i=1}^3 z_i^2(1, t) \leq 0. \quad (32)$$

Integrating (32) in $[0, s]$, for $0 \leq s \leq T$, we get

$$E(s) + \int_0^s (u_t^2(L, t) + v_t^2(L, t) + w_{tx}^2(L, t)) dt + \sum_{i=1}^3 \int_0^s z_i^2(1, t) dt \leq E(0),$$

and (28) is obtained. Taking $s = T$ and since $E(t)$ is a non-increasing function, thanks to Proposition 2.1, estimate (29) holds.

Now, we multiply the first equation of system (1) by $(T - t)u_t$, the second one by $(T - t)v_t$, the third one by $(T - t)w_t$, and integrate by parts on $(0, L) \times (0, T)$, we obtain

$$\begin{aligned} \frac{T}{2} \|U(0)\|_{\mathcal{H}}^2 &= \frac{1}{2} \int_0^T \|U(t)\|_{\mathcal{H}}^2 dt + \int_0^T (T - t) (a_1(t) \|u_t\|^2 + a_2(t) \|v_t\|^2 + a_3(t) \|w_t\|^2) dt \\ &\quad - E_1 h_1 \int_0^T (T - t) u_x(L, t) u_t(L, t) dt - E_3 h_3 \int_0^T (T - t) v_x(L, t) v_t(L, t) dt \\ &\quad - EI \int_0^T (T - t) w_{xx}(L, t) w_{tx}(L, t) dt. \end{aligned} \quad (33)$$

Since

$$\begin{aligned} -E_1 h_1 \int_0^T (T - t) u_x(L, t) u_t(L, t) dt &\leq T E_1 h_1 \left(\alpha_1 + \frac{|\beta_1|}{2} \right) \|u_t(L, \cdot)\|_{L^2(0, T)}^2 \\ &\quad + \frac{T}{2} |\beta_1| \|z_1(1, \cdot)\|_{L^2(0, T)}^2, \\ -E_3 h_3 \int_0^T (T - t) v_x(L, t) v_t(L, t) dt &\leq T E_3 h_3 \left(\alpha_2 + \frac{|\beta_2|}{2} \right) \|v_t(L, \cdot)\|_{L^2(0, T)}^2 \\ &\quad + \frac{T}{2} |\beta_2| \|z_2(1, \cdot)\|_{L^2(0, T)}^2, \end{aligned}$$

and

$$\begin{aligned} -EI \int_0^T (T - t) w_{xx}(L, t) w_{tx}(L, t) dt &\leq T EI \left(\alpha_3 + \frac{|\beta_3|}{2} \right) \|w_{tx}(L, \cdot)\|_{L^2(0, T)}^2 \\ &\quad + \frac{T}{2} |\beta_3| \|z_3(1, \cdot)\|_{L^2(0, T)}^2, \end{aligned}$$

thanks to (33) and the inequalities above, we have (30). The proof of estimate (31) follows analogously to that in [36], and the details are therefore omitted. Hence, the proof of Proposition 2.2 is complete. $\square$

3. Stabilization result

Let us prove the first main result of our work, that is, system (1) decays exponentially, giving the answers to Problem P_2 .

Proof of Theorem 1.1. Consider the following Lyapunov functional

$$\mathcal{L}(t) = \mu E(t) + \sum_{i=0}^3 \mu_i \mathcal{L}_i(t),$$

where $\mu_i \in \mathbb{R}^+$ will be chosen later. Here, $E(t)$ is the total energy given by (7), while

$$\mathcal{L}_0(t) = \rho_1 h_1 \int_0^L uu_t dx + \rho_3 h_3 \int_0^L vv_t dx + \rho h \int_0^L ww_t dx$$

and

$$\mathcal{L}_i(t) = \frac{|\beta_i|}{2} \tau_i(t) \int_0^1 (1 - \rho) z_i^2(\rho, t) d\rho, \quad i = 1, 2, 3.$$

Observe that, using the Poincaré inequality, we get

$$\begin{aligned} |\mathcal{L}_0(t)| &\leq \frac{1}{2} \left(\rho_1 h_1 \|u_t\|^2 + \rho_1 h_1 \|u\|^2 + \rho_3 h_3 \|v_t\|^2 + \rho_3 h_3 \|v\|^2 + \rho h \|w_t\|^2 + \rho h \|w\|^2 \right) \\ &\leq \frac{1}{2} \left(\rho_1 h_1 \|u_t\|^2 + \frac{\rho_1 L^2}{E_1 \pi^2} E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|v_t\|^2 + \frac{\rho_3 L^2}{E_3 \pi^2} E_3 h_3 \|v_x\|^2 + \rho h \|w_t\|^2 \right. \\ &\quad \left. + \frac{\rho h L^4}{EI \pi^4} EI \|w_{xx}\|^2 + k \|-u + v + \alpha w_x\|^2 \right). \end{aligned}$$

Since

$$|\mathcal{L}_i(t)| \leq \frac{|\beta_i|}{2} \tau_i(t) \|z_i^2(\cdot, t)\|^2,$$

we obtain that

$$(\mu - \mu_4)E(t) \leq \mathcal{L}(t) \leq (\mu + \mu_4)E(t). \quad (34)$$

Here,

$$\mu_4 = \max \left\{ \mu_0, \frac{\mu_0 \rho_1 L^2}{E_1 \pi^2}, \frac{\mu_0 \rho_3 L^2}{E_3 \pi^2}, \frac{\mu_0 \rho_1 L^2}{E_1 \pi^2}, \frac{\mu_0 \rho h L^4}{EI \pi^4}, \mu_1, \mu_2, \mu_3 \right\} \quad (35)$$

and, initially, we will take $\mu_i > 0$, $i = 0, 1, 2, 3$, small enough such that $\mu_4 < \mu$.

On the other hand, using system (1) and the boundary condition, we get that

$$\begin{aligned} \mathcal{L}'_0(t) &= \rho_1 h_1 \int_0^L uu_{tt} dx + \rho_3 h_3 \int_0^L vv_{tt} dx + \rho h \int_0^L ww_{tt} dx \\ &\quad + \rho_1 h_1 \int_0^L u_t^2 dx + \rho_3 h_3 \int_0^L v_t^2 dx + \rho h \int_0^L w_t^2 dx \\ &= \rho_1 h_1 \|u_t\|^2 - E_1 h_1 \|u_x\|^2 + \rho_3 h_3 \|v_t\|^2 - E_3 h_3 \|v_x\|^2 + \rho h \|w_t\|^2 \\ &\quad - EI \|w_{xx}\|^2 - k \|-u + v + \alpha w_x\|^2 + E_1 h_1 u_x(L, t) u(L, t) \\ &\quad + E_3 h_3 v_x(L, t) v(L, t) + EI w_{xx}(L, t) w_x(L, t) \\ &\quad - a_1(t) \int_0^L uu_t dx - a_2(t) \int_0^L vv_t dx - a_3(t) \int_0^L ww_t dx. \end{aligned} \quad (36)$$

Applying Young's inequality, the Sobolev embedding, and using the boundary conditions of system (1), the following estimate holds

$$\begin{aligned} |E_1 h_1 u_x(L, t) u(L, t)| &\leq E_1 h_1 \epsilon_1 \|u_x\|^2 + C_{\epsilon_1, E_1, h_1} |u_x(L, t)|^2 \\ &\leq E_1 h_1 \epsilon_1 \|u_x\|^2 + C_{\epsilon_1, E_1, h_1} \begin{pmatrix} u_t(L, t) \\ z_1(1, t) \end{pmatrix}^\top \begin{pmatrix} \alpha_1^2 & \alpha_1 \beta_1 \\ \alpha_1 \beta_1 & \beta_1^2 \end{pmatrix} \begin{pmatrix} u_t(L, t) \\ z_1(1, t) \end{pmatrix}, \end{aligned} \quad (37)$$

for any $\epsilon_1 > 0$ and some positive constant C_{ϵ_1, E_1, h_1} .

Similarly, for any $\epsilon_2, \epsilon_3 > 0$, we deduce that

$$|E_3 h_3 v_x(L, t) v(L, t)| \leq E_3 h_3 \epsilon_2 \|v_x\|^2 + C_{\epsilon_2, E_3, h_3} \begin{pmatrix} v_t(L, t) \\ z_2(1, t) \end{pmatrix}^\top \begin{pmatrix} \alpha_2^2 & \alpha_2 \beta_2 \\ \alpha_2 \beta_2 & \beta_2^2 \end{pmatrix} \begin{pmatrix} v_t(L, t) \\ z_2(1, t) \end{pmatrix} \quad (38)$$

and

$$|EI w_{xx}(L, t) w_x(L, t)| \leq EI \epsilon_3 \|w_{xx}\|^2 + C_{\epsilon_3, E, I} \begin{pmatrix} w_{tx}(L, t) \\ z_3(1, t) \end{pmatrix}^\top \begin{pmatrix} \alpha_3^2 & \alpha_3 \beta_3 \\ \alpha_3 \beta_3 & \beta_3^2 \end{pmatrix} \begin{pmatrix} w_{tx}(L, t) \\ z_3(1, t) \end{pmatrix}. \quad (39)$$

Additionally to that, thanks to (13)–(15), and using integration by parts, we obtain that

$$\mathcal{L}'_1(t) = -\frac{|\beta_1|}{2} \int_0^1 (1 - \dot{\tau}_1(t)\rho) z_1^2(\rho, t) d\rho + \frac{|\beta_1|}{2} u_t^2(L, t), \quad (40)$$

$$\mathcal{L}'_2(t) = -\frac{|\beta_2|}{2} \int_0^1 (1 - \dot{\tau}_2(t)\rho) z_2^2(\rho, t) d\rho + \frac{|\beta_2|}{2} v_t^2(L, t), \quad (41)$$

and

$$\mathcal{L}'_3(t) = -\frac{|\beta_3|}{2} \int_0^1 (1 - \dot{\tau}_3(t)\rho) z_3^2(\rho, t) d\rho + \frac{|\beta_3|}{2} w_{tx}^2(L, t). \quad (42)$$

Moreover, using (3) and Young's and Poincaré's inequalities, it follows that

$$\begin{aligned} -a_1(t) \int_0^L uu_t dx &\leq \widehat{\epsilon}_1 b_{01}^2 \|u\|^2 + C_{\widehat{\epsilon}_1} \|u_t\|^2 \\ &\leq \widehat{\epsilon}_1 b_{01}^2 \frac{L^2}{\pi^2} \|u_x\|^2 + C_{\widehat{\epsilon}_1} \|u_t\|^2, \end{aligned} \quad (43)$$

$$-a_2(t) \int_0^L vv_t dx \leq \widehat{\epsilon}_2 b_{02}^2 \frac{L^2}{\pi^2} \|v_x\|^2 + C_{\widehat{\epsilon}_2} \|v_t\|^2, \quad (44)$$

and

$$-a_3(t) \int_0^L ww_t dx \leq \widehat{\epsilon}_3 b_{03}^2 \frac{L^4}{\pi^4} \|w_{xx}\|^2 + C_{\widehat{\epsilon}_3} \|w_t\|^2 \quad (45)$$

for any $\widehat{\epsilon}_i > 0$ and some constants $C_{\widehat{\epsilon}_i} > 0$, $i = 1, 2, 3$.

Thus, using (3), (25) and (36)–(45), it follows that

$$\begin{aligned} \mathcal{L}'(t) &= \mu E'(t) + \sum_{i=0}^3 \mu_i \mathcal{L}'_i(t) \\ &\leq \left[-\left(\frac{\mu a_{01}}{\rho_1 h_1} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_1}}{\rho_1 h_1} \right) \right) \rho_1 h_1 \|u_t\|^2 - \mu_0 \left(1 - \left(\epsilon_1 + \frac{\widehat{\epsilon}_1 b_{01}^2 L^2}{E_1 h_1 \pi^2} \right) \right) E_1 h_1 \|u_x\|^2 \right. \\ &\quad - \left(\frac{\mu a_{02}}{\rho_3 h_3} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_2}}{\rho_3 h_3} \right) \right) \rho_3 h_3 \|v_t\|^2 - \mu_0 \left(1 - \left(\epsilon_2 + \frac{\widehat{\epsilon}_2 b_{02}^2 L^2}{E_3 h_3 \pi^2} \right) \right) E_3 h_3 \|v_x\|^2 \\ &\quad - \left(\frac{\mu a_{03}}{\rho h} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_3}}{\rho h} \right) \right) \rho h \|w_t\|^2 - \mu_0 \left(1 - \left(\epsilon_3 + \frac{\widehat{\epsilon}_3 b_{03}^2 L^4}{EI \pi^4} \right) \right) EI \|w_{xx}\|^2 \\ &\quad \left. - \mu_0 k \|-u + v + \alpha w_x\|^2 - \mu_0 \sum_{i=1}^3 \frac{|\beta_i|}{2} \tau_i(t) \|z_i^2(\cdot, t)\|_{L^2(0,1)}^2 \right] + \sum_{i=1}^3 W_i + \sum_{i=1}^3 S_i, \end{aligned} \quad (46)$$

where

$$\begin{cases} W_1 = \frac{1}{2} \left\langle \Pi_{\mu_0, \mu_1}(u_t(L, t), z_1(1, t)), (u_t(L, t), z_1(1, t)) \right\rangle, \\ \Pi_{\mu_0, \mu_1} = \mu \Phi_{\alpha_1, \beta_1}^{\dot{\tau}_1(t)} + \mu_0 C_{\epsilon_1, E_1, h_1} \begin{pmatrix} \alpha_1^2 & \alpha_1 \beta_1 \\ \alpha_1 \beta_1 & \beta_1^2 \end{pmatrix} + \mu_1 |\beta_1| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \\ W_2 = \frac{1}{2} \left\langle \Pi_{\mu_0, \mu_2}(v_t(L, t), z_2(1, t)), (v_t(L, t), z_2(1, t)) \right\rangle, \\ \Pi_{\mu_0, \mu_2} = \mu \Phi_{\alpha_2, \beta_2}^{\dot{\tau}_2(t)} + \mu_0 C_{\epsilon_2, E_2, h_2} \begin{pmatrix} \alpha_2^2 & \alpha_2 \beta_2 \\ \alpha_2 \beta_2 & \beta_2^2 \end{pmatrix} + \mu_2 |\beta_2| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \\ W_3 = \frac{1}{2} \left\langle \Pi_{\mu_0, \mu_3}(w_{tx}(L, t), z_3(1, t)), (w_{tx}(L, t), z_3(1, t)) \right\rangle, \\ \Pi_{\mu_0, \mu_3} = \mu \Phi_{\alpha_3, \beta_3}^{\dot{\tau}_3(t)} + \mu_0 C_{\epsilon_3, E, I} \begin{pmatrix} \alpha_3^2 & \alpha_3 \beta_3 \\ \alpha_3 \beta_3 & \beta_3^2 \end{pmatrix} + \mu_3 |\beta_3| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases}$$

and

$$S_i = -\mu_i \frac{|\beta_i|}{2} \int_0^1 (1 - \dot{\tau}_i(t) \rho) z_i^2(\rho, t) d\rho + \mu_0 \frac{|\beta_i|}{2} \tau_i(t) \|z_i^2(\cdot, t)\|_{L^2(0,1)}^2, \quad i = 1, 2, 3.$$

Our task now is to prove that

$$\mathcal{L}'(t) \leq -\lambda E(t),$$

for some positive constant λ . To do so, let us analyze each term W_i and S_i in (46), for $i = 1, 2, 3$.

Estimate for W_i : Since the matrices $\Phi_{\alpha_i, \beta_i}^{\dot{\tau}_i(t)}$ are negative definite, it follows from the continuity of the trace and determinant functions that one can choose $\mu_0, \mu_i \in (0, 1)$ sufficiently small so that the perturbed matrix Π_{μ_0, μ_i} remains negative definite. Thus,

$$W_i \leq 0, \text{ for } i = 1, 2, 3. \quad (47)$$

Estimate for S_i : By using (2), we get

$$\begin{aligned} S_i &\leq -\frac{\mu_i |\beta_i|}{2} (1 - d_i) \|z_i^2(\cdot, t)\|_{L^2(0,1)}^2 + \mu_0 \frac{|\beta_i|}{2} M_i \|z_i^2(\cdot, t)\|_{L^2(0,1)}^2 \\ &\leq -(\mu_i (1 - d_i) - \mu_0 M_i) \frac{|\beta_i|}{2} \|z_i^2(\cdot, t)\|_{L^2(0,1)}^2. \end{aligned}$$

So, choosing

$$\frac{\mu_0 M_i}{(1 - d_i)} < \mu_i,$$

we have that

$$S_i < 0, \quad i = 1, 2, 3. \quad (48)$$

Thus, let us choose $\epsilon_i, \widehat{\epsilon}_i > 0, i = 1, 2, 3, \mu_0 > 0$ sufficiently small and $\mu > 0$ such that

$$\max \left\{ \epsilon_1 + \frac{\widehat{\epsilon}_1 b_{01}^2 L^2}{E_1 h_1 \pi^2}, \epsilon_2 + \frac{\widehat{\epsilon}_2 b_{02}^2 L^2}{E_3 h_3 \pi^2}, \epsilon_3 + \frac{\widehat{\epsilon}_3 b_{03}^2 L^4}{EI \pi^4} \right\} < 1,$$

$$\frac{\mu_0 M_i}{1 - d_i} < \mu_i, \quad i = 1, 2, 3,$$

and

$$\max \{\mu_4, \mu_5\} < \mu,$$

where

$$\mu_5 = \max \left\{ \frac{\mu_0}{a_{01}}(\rho_1 h_1 + C_{\widehat{\epsilon}_1}), \frac{\mu_0}{a_{02}}(\rho_3 h_3 + C_{\widehat{\epsilon}_2}), \frac{\mu_0}{a_{03}}(\rho h + C_{\widehat{\epsilon}_3}) \right\}$$

and μ_4 is given by (35).

Next, consider $\lambda > 0$ satisfying

$$\lambda \leq \min \left\{ 2 \left(\frac{\mu a_{01}}{\rho_1 h_1} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_1}}{\rho_1 h_1} \right) \right), 2 \left(\frac{\mu a_{02}}{\rho_3 h_3} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_2}}{\rho_3 h_3} \right) \right), 2 \left(\frac{\mu a_{03}}{\rho h} - \mu_0 \left(1 + \frac{C_{\widehat{\epsilon}_3}}{\rho h} \right) \right), \right. \\ \left. 2\mu_0 \left(1 - \left(\epsilon_1 + \frac{\widehat{\epsilon}_1 b_{01}^2 L^2}{E_1 h_1 \pi^2} \right) \right), 2\mu_0 \left(1 - \left(\epsilon_2 + \frac{\widehat{\epsilon}_2 b_{02}^2 L^2}{E_3 h_3 \pi^2} \right) \right), 2\mu_0 \left(1 - \left(\epsilon_3 + \frac{\widehat{\epsilon}_3 b_{03}^2 L^4}{EI \pi^4} \right) \right) \right\}.$$

Then, from (46)–(48) and using inequality (34), it follows that $\lambda > 0$ satisfies

$$\mathcal{L}'(t) \leq -\lambda E(t) \leq -\frac{\lambda}{\mu_*} \mathcal{L}(t),$$

where $\mu_* = \mu + \mu_4$, and consequently,

$$\mathcal{L}(t) \leq e^{-\frac{\lambda}{\mu_*} t} \mathcal{L}(0). \quad (49)$$

Finally, combining (34) and (49), there exist positive constants λ , $\zeta = \frac{\mu + \mu_4}{\mu - \mu_4}$ and $\mu_* = \mu + \mu_4$ such that

$$E(t) \leq \zeta e^{-\frac{\lambda}{\mu_*} t} E(0), \quad \text{for all } t \geq 0.$$

This completes the proof of Theorem 1.1. □

4. Well-posedness for the control system

In this section, we establish the well-posedness of system (8). We consider both the homogeneous case, where the boundary inputs f_1 , f_2 , and f_3 vanish, and the nonhomogeneous case, in which these three controls are present.

4.1. The homogeneous system

We begin by considering the following homogeneous system:

$$\left\{ \begin{array}{ll} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, & x \in (0, L), \quad t > 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, & x \in (0, L), \quad t > 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, & x \in (0, L), \quad t > 0, \\ u(0, t) = v(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0, & t > 0, \\ w_{xx}(L, t) = w_{xxx}(L, t) = 0, & t > 0, \\ u_{tt}(L, t) + u_x(L, t) = 0, & t > 0, \\ v_{tt}(L, t) + v_x(L, t) = 0, & t > 0, \\ w_{tt}(L, t) - u(L, t) + v(L, t) + \alpha w_x(L, t) = 0, & t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in (0, L), \\ (u_t, v_t, w_t)(x, 0) = (u_1, v_1, w_1)(x), & x \in (0, L). \end{array} \right. \quad (50)$$

Observe that, by a formal computation, we find that a solution of system (50) satisfies the following identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} & \left[\rho_1 h_1 \|u_t\|_{L^2}^2 + E_1 h_1 \|u_x\|_{L^2}^2 + \rho_3 h_3 \|v_t\|_{L^2}^2 + E_3 h_3 \|v_x\|_{L^2}^2 + \rho h \|w_t\|_{L^2}^2 + EI \|w_{xx}\|_{L^2}^2 \right. \\ & \left. + k \|-u + v + \alpha w_x\|_{L^2}^2 + E_1 h_1 |u_t(L)|^2 + E_3 h_3 |v_t(L)|^2 + \alpha k |w_t(L)|^2 \right] = 0. \end{aligned}$$

With this in mind, we introduce the new variables

$$\Psi_1 = u_t, \quad \Psi_2 = v_t, \quad \Psi_3 = w_t, \quad \Psi_4(\cdot) = \Psi_1(L, \cdot), \quad \Psi_5(\cdot) = \Psi_2(L, \cdot), \quad \Psi_6(\cdot) = \Psi_3(L, \cdot),$$

and define the vector functions

$$U = (u, \Psi_1, v, \Psi_2, w, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \quad \text{and} \quad U_0 = (u_0, u_1, v_0, v_1, w_0, w_1, \Psi_4(0), \Psi_5(0), \Psi_6(0))^\top.$$

Hence, system (50) can be reformulated as an abstract Cauchy problem

$$\begin{cases} U_t = \mathcal{P}U \\ U(0) = U_0, \end{cases} \quad (51)$$

where the operator $\mathcal{P}$ is given by

$$\mathcal{P}U := \begin{pmatrix} \Psi_1 \\ \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x)] \\ \Psi_2 \\ \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x)] \\ \Psi_3 \\ \frac{1}{\rho h} [-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x] \\ -u_x(L, \cdot) \\ -v_x(L, \cdot) \\ -[-u(L, \cdot) + v(L, \cdot) + \alpha w_x(L, \cdot)] \end{pmatrix}. \quad (52)$$

Furthermore, the domain of $\mathcal{P} : D(\mathcal{P}) \subset \mathcal{H}_2 \rightarrow \mathcal{H}_2$ is defined by

$$D(\mathcal{P}) = \left\{ \begin{array}{l} U = (u, \Psi_1, v, \Psi_2, w, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in \mathcal{H}_2; \\ (u, v, w) \in [H^2(0, L) \cap H_*^1(0, L)]^2 \times H_*^4(0, L), \\ (\Psi_1, \Psi_2, \Psi_3) \in [H_*^1(0, L)]^2 \times H_*^2(0, L), \\ \Psi_4 = \Psi_1(L), \Psi_5 = \Psi_2(L), \Psi_6 = \Psi_3(L) \end{array} \right\}. \quad (53)$$

Now, let us denote by $\rho(\mathcal{P})$ the resolvent set of the operator $\mathcal{P}$. Then, we have the following result.

Lemma 4.1. *Let $\mathcal{H}_2$ and $(D(\mathcal{P}), \mathcal{P})$ be defined as before. Then, $0 \in \rho(\mathcal{P})$. Moreover, $\mathcal{P}^{-1}$ is compact.*

Proof. For $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9) \in \mathcal{H}_2$, we show the existence of $U \in D(\mathcal{P})$, a unique solution of the equation

$$\mathcal{P}U = G.$$

Equivalently, one must consider the system given by

$$\Psi_1 = g_1, \quad \Psi_2 = g_3, \quad \Psi_3 = g_5, \quad (54)$$

$$E_1 h_1 u_{xx} + k(-u + v + \alpha w_x) = \rho_1 h_1 g_2, \quad (55)$$

$$E_3 h_3 v_{xx} - k(-u + v + \alpha w_x) = \rho_3 h_3 g_4, \quad (56)$$

$$EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x = \rho h g_6, \quad (57)$$

$$-u_x(L) = g_7, \quad (58)$$

$$-v_x(L) = g_8, \quad (59)$$

$$-[-u(L) + v(L) + \alpha w_x(L)] = g_9, \quad (60)$$

with the following boundary conditions

$$u(0) = v(0) = w_x(0) = w_{xxx}(0) = w_{xx}(L) = w_{xxx}(L) = 0. \quad (61)$$

Let $(u^*, v^*, w^*) \in [H_*^1(0, L)]^2 \times H_*^2(0, L)$. We multiply the equations (55), (56), and (57) by u^* , v^* , and w^* , respectively, and integrate by parts over $(0, L)$. Using (58), (59), (60), and the boundary conditions in (61), and then summing the resulting expressions, we obtain

$$\mathcal{B}((u, v, w), (u^*, v^*, w^*)) = \mathcal{L}(u^*, v^*, w^*), \quad \forall (u^*, v^*, w^*) \in [H_*^1(0, L)]^2 \times H_*^2(0, L), \quad (62)$$

where

$$\begin{aligned} \mathcal{B}((u, v, w), (u^*, v^*, w^*)) &= E_1 h_1 \langle u_x, u_x^* \rangle + E_3 h_3 \langle v_x, v_x^* \rangle + EI \langle w_{xx}, w_{xx}^* \rangle \\ &\quad + k \langle -u + v + \alpha w_x, -u^* + v^* + \alpha w_x^* \rangle \end{aligned} \quad (63)$$

and

$$\begin{aligned} \mathcal{L}(u^*, v^*, w^*) &= -\rho_1 h_1 \langle g_2, u^* \rangle - \rho_3 h_3 \langle g_4, v^* \rangle - \rho h \langle g_6, w^* \rangle \\ &\quad - E_1 h_1 g_7 u^*(L) - E_3 h_3 g_8 v^*(L) - \alpha k g_9 w^*(L). \end{aligned} \quad (64)$$

From (63) and (64), it follows that the bilinear form

$$\mathcal{B} : \left[\left[H_*^1(0, L) \right]^2 \times H_*^2(0, L) \right]^2 \rightarrow \mathbb{C}$$

is continuous and coercive, and the linear functional

$$\mathcal{L} : \left[H_*^1(0, L) \right]^2 \times H_*^2(0, L) \rightarrow \mathbb{C}$$

is continuous as well. Therefore, by the Lax–Milgram theorem, there exists a unique weak solution

$$(u, v, w) \in \left[H_*^1(0, L) \right]^2 \times H_*^2(0, L)$$

to the variational problem (62).

Furthermore, using (54) and standard elliptic regularity results, we conclude that $U \in D(\mathcal{P})$, which shows that the operator $\mathcal{P}$ is bijective. Following arguments similar to [39, Proposition 2], one can also verify that $\mathcal{P}^{-1}$ is bounded, implying that $0 \in \rho(\mathcal{P})$, the resolvent set of $\mathcal{P}$. Finally, by the Sobolev embedding theorem, the natural inclusion

$$i : D(\mathcal{P}) \hookrightarrow \mathcal{H}_2$$

is compact, which ensures that $\mathcal{P}^{-1}$ is a compact operator. $\square$

The next result ensures that the operator $\mathcal{P}$ generates a group. Precisely, we have the following.

Proposition 4.1. *The operator $(D(\mathcal{P}), \mathcal{P})$ is the infinitesimal generator of a group of isometries $\{S(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}_2$.*

Proof. Let $U^\# \in D(\mathcal{P})$ given by (10). Then, for all $U \in D(\mathcal{P})$, given by (9), we obtain, after some integration by parts and rearrangements, that

$$\begin{aligned} \langle \mathcal{P}U, U^\# \rangle_{\mathcal{H}_2} &= \rho_1 h_1 \left\langle \frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx} + k(-u + v + \alpha w_x)], \Psi_1^\# \right\rangle + E_1 h_1 \langle \Psi_{1x}, u_x^\# \rangle \\ &\quad + \rho_3 h_3 \left\langle \frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx} - k(-u + v + \alpha w_x)], \Psi_2^\# \right\rangle + E_3 h_3 \langle \Psi_{2x}, v_x^\# \rangle \\ &\quad + \rho h \left\langle \frac{1}{\rho h} [-EI w_{xxxx} + \alpha k(-u + v + \alpha w_x)_x], \Psi_3^\# \right\rangle + EI \langle \Psi_{3xx}, w_{xx}^\# \rangle \\ &\quad + k \langle -\Psi_1 + \Psi_2 + \alpha \Psi_{3x}, -u^\# + v^\# + \alpha w_x^\# \rangle - E_1 h_1 (u_x(L), \Psi_4^\#)_{\mathbb{R}} \\ &\quad - E_3 h_3 (v_x(L), \Psi_5^\#)_{\mathbb{R}} - \alpha k ([-u(L) + v(L) + \alpha k w_x(L)], \Psi_6^\#)_{\mathbb{R}} \\ &= \rho_1 h_1 \left\langle \Psi_1, -\frac{1}{\rho_1 h_1} [E_1 h_1 u_{xx}^\# + k(-u^\# + v^\# + \alpha w_x^\#)] \right\rangle + E_1 h_1 \langle u_x, -\Psi_{1x}^\# \rangle \\ &\quad + \rho_3 h_3 \left\langle \Psi_2, -\frac{1}{\rho_3 h_3} [E_3 h_3 v_{xx}^\# - k(-u^\# + v^\# + \alpha w_x^\#)] \right\rangle + E_3 h_3 \langle v_x, -\Psi_{2x}^\# \rangle \\ &\quad + \rho h \left\langle \Psi_3, -\frac{1}{\rho h} [-EI w_{xxxx}^\# + \alpha k(-u^\# + v^\# + \alpha w_x^\#)_x] \right\rangle + EI \langle w_{xx}, -\Psi_{3xx}^\# \rangle \\ &\quad + k \langle -u + v + \alpha w_x, -[\Psi_1^\# + \Psi_2^\# + \alpha \Psi_{3x}^\#] \rangle + E_1 h_1 (\Psi_4, u_x^\#(L))_{\mathbb{R}} \\ &\quad + E_3 h_3 (\Psi_5, v_x^\#(L))_{\mathbb{R}} + \alpha k (\Psi_6, [-u^\#(L) + v^\#(L) + \alpha k w_x^\#(L)])_{\mathbb{R}} \\ &= \langle U, -\mathcal{P}U^\# \rangle_{\mathcal{H}_2}. \end{aligned}$$

Hence, $D(\mathcal{P}) \subset D(\mathcal{P}^*)$ and $\mathcal{P}^*U^\sharp = -\mathcal{P}U^\sharp$, for any $U^\sharp \in D(\mathcal{P})$. On the other hand, let $U^\sharp \in D(\mathcal{P}^*)$. Then, by definition,

$$\langle \mathcal{P}U, U^\sharp \rangle_{\mathcal{H}_2} = \langle U, \mathcal{P}^*U^\sharp \rangle_{\mathcal{H}_2}, \text{ for all } U \in D(\mathcal{P}). \quad (65)$$

By the same calculations as above, we have that

$$\langle U, \mathcal{P}^*U^\sharp \rangle_{\mathcal{H}_2} = \langle \mathcal{P}U, U^\sharp \rangle_{\mathcal{H}_2}. \quad (66)$$

Therefore, from (65) and (66), we obtain that

$$\begin{aligned} 0 = & -E_1 h_1 \left(u_x(0), \Psi_1^\sharp(0) \right)_\mathbb{R} - E_3 h_3 \left(v_x(0), \Psi_2^\sharp(0) \right)_\mathbb{R} - EI \left(w_{xx}(0), \Psi_{3x}^\sharp(0) \right)_\mathbb{R} \\ & + E_1 h_1 \left\langle u_x(L), \Psi_1^\sharp(L) - \Psi_4^\sharp \right\rangle + E_3 h_3 \left\langle v_x(L), \Psi_2^\sharp(L) - \Psi_5^\sharp \right\rangle \\ & - EI \left(w_{xx}(0), \Psi_{3x}^\sharp(0) \right)_\mathbb{R} + EI \left\langle \Psi_{3x}(L), w_{xx}^\sharp(L) \right\rangle - EI \left\langle \Psi_3(L), w_{xxx}^\sharp(L) \right\rangle \\ & + \alpha k \left([-u(L) + v(L) + \alpha k w_x(L)], \Psi_3^\sharp(L) - \Psi_6^\sharp \right)_\mathbb{R}. \end{aligned}$$

Since the above equality holds for all $U \in D(\mathcal{P})$, it follows that $U^\sharp \in D(\mathcal{P})$. Consequently, we deduce that $D(\mathcal{P}) = D(\mathcal{P}^*)$, and from the previous computations, we also obtain $\mathcal{P}^* = -\mathcal{P}$. Therefore, $\mathcal{P}$ is a skew-adjoint operator, and by Stone's Theorem, it generates a group of isometries $\{S(t)\}_{t \in \mathbb{R}}$ on $\mathcal{H}_2$. $\square$

As an immediate consequence of Lemma 4.1, Proposition 4.1, and standard results from the theory of evolution equations (see, e.g., [38]), we obtain the following existence and uniqueness result.

Theorem 4.1. *For any $U_0 \in D(\mathcal{P})$, there exists a unique solution U of the abstract Cauchy problem (51) such that*

$$U \in C([0, \infty); D(\mathcal{P})) \cap C^1([0, \infty); \mathcal{H}_2). \quad (67)$$

Moreover, for any $t > 0$, we have that

$$\|U(t)\|_{\mathcal{H}_2}^2 = \|U_0\|_{\mathcal{H}_2}^2. \quad (68)$$

4.2. The nonhomogeneous system

In this subsection, we focus on analyzing the complete system (8). We start with the following result:

Proposition 4.2. *For any $U_0 \in D(\mathcal{P})$ and $f_i \in C_0^\infty(0, \infty)$, $i = 1, 2, 3$, system (8) has a unique solution $U \in C([0, \infty); D(\mathcal{P})) \cap C^1([0, \infty); \mathcal{H}_2)$.*

Proof. Let $\phi_1 \in C^\infty[0, L]$ be a smooth function satisfying the boundary conditions $\phi_1(0) = \phi_1(L) = 0$ and $\phi_1'(L) = -1$. We now introduce the following change of variables:

$$\begin{pmatrix} z \\ \eta \\ \varphi \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} f_1(t)\phi_1(x) \\ f_2(t)\phi_1(x) \\ f_3(t)\phi_1(x) \end{pmatrix}, \quad (69)$$

where $(\tilde{u}, \tilde{v}, \tilde{w})$ is the unique solution of the system

$$\left\{ \begin{array}{ll} \rho_1 h_1 \tilde{u}_{tt} - E_1 h_1 \tilde{u}_{xx} - k(-\tilde{u} + \tilde{v} + \alpha \tilde{w}_x) = 0, & x \in (0, L), \quad t > 0, \\ \rho_3 h_3 \tilde{v}_{tt} - E_3 h_3 \tilde{v}_{xx} + k(-\tilde{u} + \tilde{v} + \alpha \tilde{w}_x) = 0, & x \in (0, L), \quad t > 0, \\ \rho h \tilde{w}_{tt} + EI \tilde{w}_{xxxx} - \alpha k(-\tilde{u} + \tilde{v} + \alpha \tilde{w}_x)_x = 0, & x \in (0, L), \quad t > 0, \\ \tilde{u}(0, t) = \tilde{v}(0, t) = \tilde{w}_x(0, t) = \tilde{w}_{xxx}(0, t) = 0, & t > 0, \\ \tilde{w}_{xx}(L, t) = \tilde{w}_{xxx}(L, t) = 0, & t > 0, \\ \tilde{u}_{tt}(L, t) + \tilde{u}_x(L, t) = 0, & t > 0, \\ \tilde{v}_{tt}(L, t) + \tilde{v}_x(L, t) = 0, & t > 0, \\ \tilde{w}_{tt}(L, t) - \tilde{u}(L, t) + \tilde{v}(L, t) + \alpha \tilde{w}_x(L, t) = 0, & t > 0, \\ (\tilde{u}, \tilde{v}, \tilde{w})(x, 0) = (u_0, v_0, w_0)(x), & x \in (0, L), \\ (\tilde{u}_t, \tilde{v}_t, \tilde{w}_t)(x, 0) = (u_1, v_1, w_1)(x), & x \in (0, L), \end{array} \right. \quad (70)$$

given by Theorem 4.1. Thus, (z, η, φ) solves the following problem

$$\left\{ \begin{array}{ll} \rho_1 h_1 z_{tt} - E_1 h_1 z_{xx} - k(-z + \eta + \alpha \varphi_x) = F, & x \in (0, L), \quad t > 0, \\ \rho_3 h_3 \eta_{tt} - E_3 h_3 \eta_{xx} + k(-z + \eta + \alpha \varphi_x) = G, & x \in (0, L), \quad t > 0, \\ \rho h \varphi_{tt} + EI \varphi_{xxxx} - \alpha k(-z + \eta + \alpha \varphi_x)_x = H, & x \in (0, L), \quad t > 0, \\ z(0, t) = \eta(0, t) = \varphi_x(0, t) = \varphi_{xxx}(0, t) = 0, & t > 0, \\ \varphi_{xx}(L, t) = \varphi_{xxx}(L, t) = 0, & t > 0, \\ z_{tt}(L, t) + z_x(L, t) = 0, & t > 0, \\ \eta_{tt}(L, t) + \eta_x(L, t) = 0, & t > 0, \\ \varphi_{tt}(L, t) - z(L, t) + \eta(L, t) + \alpha \varphi_x(L, t) = 0, & t > 0, \\ (z, \eta, \varphi)(x, 0) = (0, 0, 0), & x \in (0, L), \\ (z_t, \eta_t, \varphi_t)(x, 0) = (0, 0, 0), & x \in (0, L). \end{array} \right. \quad (71)$$

Here,

$$\left\{ \begin{array}{l} F = \rho_1 h_1 f_1''(t) \phi_1(x) - E_1 h_1 f_1(t) \phi_1''(x) - k[-f_1(t) \phi_1(x) + f_2(t) \phi_1(x) + \alpha f_3(t) \phi_1'(x)], \\ G = \rho_3 h_3 f_2''(t) \phi_1(x) - E_3 h_3 f_2(t) \phi_1''(x) + k[-f_1(t) \phi_1(x) + f_2(t) \phi_1(x) + \alpha f_3(t) \phi_1'(x)], \\ H = \rho h f_3''(t) \phi_1(x) + EI f_3(t) \phi_1^{(4)}(x) - \alpha k[-f_1(t) \phi_1'(x) + f_2(t) \phi_1'(x) + \alpha f_3(t) \phi_1''(x)], \end{array} \right.$$

and

$$(F(t, x), G(t, x), H(t, x)) \in [C^1([0, \infty) \times [0, L])]^3.$$

Now, let us use the notation established in the previous section to express the system (71) as an abstract evolution equation in the following form

$$\left\{ \begin{array}{l} W_t + \mathcal{P}W = \mathcal{K}, \\ W(0) = 0, \end{array} \right.$$

where $W = (z, z_t, \eta, \eta_t, \varphi, \varphi_t, z_t(L, T), \eta_t(L, t), \varphi_t(L, t))^T$ and

$$\mathcal{K} = \left(0, \frac{F}{\rho_1 h_1}, 0, \frac{G}{\rho_3 h_3}, 0, \frac{H}{\rho h}, 0, 0, 0\right)^T \in [C^1([0, \infty] \times [0, L])]^9.$$

Since the operator $\mathcal{P}$ generates a group of isometries on the Hilbert space $\mathcal{H}_2$, it follows that system (71) admits a unique solution

$$W \in C([0, \infty); D(\mathcal{P})) \cap C^1([0, \infty); \mathcal{H}_2).$$

Therefore, by reverting the change of variables in (69), the proof is complete. $\square$

With the well-posedness result established, we now proceed to analyze solutions of system (8) in the transposition sense. For that, given $T > 0$, $U_0 \in D(\mathcal{P})$,

$$(J_1, J_2, J_3) \in L^2\left(0, T; \left([H_*^1(0, L)]^2 \times H_*^2(0, L)\right)^*\right)$$

and $f_i \in L^2(0, T)$, for $i = 1, 2, 3$, consider the non-homogeneous system

$$\left\{ \begin{array}{ll} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = J_1, & x \in (0, L), \quad t > 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = J_2, & x \in (0, L), \quad t > 0, \\ \rho h w_{tt} + E I w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = J_3, & x \in (0, L), \quad t > 0, \\ u(0, t) = v(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0, & t > 0, \\ w_{xx}(L, t) = w_{xxx}(L, t) = 0, & t > 0, \\ u_{tt}(L, t) + u_x(L, t) = f_1, & t > 0, \\ v_{tt}(L, t) + v_x(L, t) = f_2, & t > 0, \\ w_{tt}(L, t) - u(L, t) + v(L, t) + \alpha w_x(L, t) = f_3, & t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in (0, L), \\ (u_t, v_t, w_t)(x, 0) = (u_1, v_1, w_1)(x), & x \in (0, L). \end{array} \right. \quad (72)$$

Now, let us introduce the following definition.

Definition 4.1. A solution by transposition to system (72) is defined as a vector-valued function $U = (u, u_t, v, v_t, w, w_t, u_t(L), v_t(L), w_t(L))^T$ belonging to the space $C([0, T]; \mathcal{D}(\mathcal{P}))$, which satisfies the following property: For every $\tau \in [0, T]$ and every function $W^\tau \in \mathcal{D}(\mathcal{P}^*)$ of the form

$$W^\tau = (\chi_0^\tau, \chi_1^\tau, \eta_0^\tau, \eta_1^\tau, \Theta_0^\tau, \Theta_1^\tau, \chi_1^\tau(L), \eta_1^\tau(L), \Theta_1^\tau(L))^T,$$

the following integral identity holds:

$$\begin{aligned}
 & \left\langle \begin{pmatrix} \rho_1 h_1 u_t(\cdot, \tau) \\ -\rho_1 h_1 u(\cdot, \tau) \\ \rho_3 h_3 v_t(\cdot, \tau) \\ -\rho_3 h_3 v(\cdot, \tau) \\ \rho h w_t(\cdot, \tau), -\rho h w(\cdot, \tau) \\ -E_1 h_1 u(L, \tau) \\ -E_3 h_3 v(L, \tau) \\ -\alpha k w(L, \tau) \end{pmatrix}^\top, \begin{pmatrix} \chi_0^\tau \\ \chi_1^\tau \\ \eta_0^\tau \\ \eta_1^\tau \\ \Theta_0^\tau \\ \Theta_1^\tau \\ \chi_t(L, \tau) \\ \eta_t(L, \tau) \\ \Theta_t(L, \tau) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \rho_1 h_1 u_1 \\ -\rho_1 h_1 u_0 \\ \rho_3 h_3 v_1 \\ -\rho_3 h_3 v_0 \\ \rho h w_1 \\ -\rho h w_0 \\ -E_1 h_1 u(L, 0) \\ -E_3 h_3 v(L, 0) \\ -\alpha k w(L, 0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi_t(L, 0) \\ \eta_t(L, 0) \\ \Theta_t(L, 0) \end{pmatrix} \right\rangle \quad (73) \\
 & + \int_0^\tau \langle (J_1(t), J_2(t), J_3(t)), (\chi(t), \eta(t), \Theta(t)) \rangle_{[(H_*^1(0,L))^*, H_*^1(0,L)]^3} dt \\
 & + ((E_1 h_1 f_1(t), E_3 h_3 f_2(t), \alpha k f_3(t)), 1_{(0,\tau)}(\chi(L, t), \eta(L, t), \Theta(L, t)))_{[L^2(0,T)]^3} \\
 & + ((E_1 h_1 u_t(L, 0), E_3 h_3 v_t(L, 0), \alpha k w_t(L, 0), (\chi(L, 0), \eta(L, 0), \Theta(L, 0))))_{\mathbb{R}^3},
 \end{aligned}$$

where the inner product is in $\mathcal{H}_2$ and $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^\top$ is the solution of the adjoint system

$$\begin{cases} \rho_1 h_1 \chi_{tt} - E_1 h_1 \chi_{xx} - k(-\chi + \eta + \alpha \Theta_x) = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \rho_3 h_3 \eta_{tt} - E_3 h_3 \eta_{xx} + k(-\chi + \eta + \alpha \Theta_x) = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \rho h \Theta_{tt} + EI \Theta_{xxxx} - \alpha k(-\chi + \eta + \alpha \Theta_x)_x = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \chi(0, t) = \eta(0, t) = \Theta_x(0, t) = \Theta_{xxx}(0, t) = 0, & t \in (0, \tau), \\ \Theta_{xx}(L, t) = \Theta_{xxx}(L, t) = 0, & t \in (0, \tau), \\ \chi_{tt}(L, t) + \chi_x(L, t) = 0, & t \in (0, \tau), \\ \eta_{tt}(L, t) + \eta_x(L, t) = 0, & t \in (0, \tau), \\ \Theta_{tt}(L, t) - \chi(L, t) + \eta(L, t) + \alpha \Theta_x(L, t) = 0, & t \in (0, \tau), \\ (\chi, \eta, \Theta)(x, 0) = (\chi_0, \eta_0, \Theta_0)(x), & x \in (0, L), \\ (\chi_t, \eta_t, \Theta_t)(x, 0) = (\chi_1, \eta_1, \Theta_1)(x), & x \in (0, L), \\ (\chi, \eta, \Theta)(L, \tau) = (0, 0, 0). \end{cases} \quad (74)$$

The following proposition guarantees that system (8) admits a unique solution in the transposition sense.

Proposition 4.3. Let $T > 0$, $U_0 \in D(\mathcal{P})$, $(J_1, J_2, J_3) \in \left(L^2\left(0, T; \left(H_*^1(0, L)\right)^*\right)\right)^3$, and $f_i \in L^2(0, T)$, for $i = 1, 2, 3$. Then, there exists a unique solution

$$U = (u, u_t, v, v_t, w, w_t, u_t(L), v_t(L), w_t(L))^\top \in C([0, T]; D(\mathcal{P}))$$

of system (72) that verifies (73).

Proof. Let $T > 0$ and $\tau \in [0, T]$. According to Theorem 4.1, system (74) has a unique solution W expressed as

$$W(t) = S(\tau - t)W^\tau,$$

and there exists a constant $C_T > 0$ such that

$$\|W(t)\|_{\mathcal{H}_2} \leq C_T \|W^\tau\|_{\mathcal{H}_2}, \quad \forall t \in [0, \tau]. \quad (75)$$

We define $\mathcal{R}$ as the linear functional corresponding to the right-hand side of (73), namely,

$$\begin{aligned} \mathcal{R}(W^\tau) = & \left\langle V_0^1, W(0) \right\rangle_{\mathcal{H}_2} + \int_0^\tau \langle (J_1(t), J_2(t), J_3(t)), (\chi(t), \eta(t), \Theta(t)) \rangle_{[(H_*^1(0,L))^*, H_*^1(0,L)]^3} dt \\ & + ((E_1 h_1 f_1(t), E_3 h_3 f_2(t), \alpha k f_3(t)), 1_{(0,\tau)}(\chi(L, t), \eta(L, t), \Theta(L, t)))_{[L^2(0,T)]^3} \\ & + ((E_1 h_1 u_t(L, 0), E_3 h_3 v_t(L, 0), \alpha k w_t(L, 0), (\chi(L, 0), \eta(L, 0), \Theta(L, 0))))_{\mathbb{R}^3}, \end{aligned}$$

where

$$\begin{aligned} V_0^1 = & \left(\rho_1 h_1 u_1, -\rho_1 h_1 u_0, \rho_3 h_3 v_1, -\rho_3 h_3 v_0, \rho h w_1, \right. \\ & \left. -\rho h w_0, -E_1 h_1 u(L, 0), -E_3 h_3 v(L, 0), -\alpha k w(L, 0) \right)^\top. \end{aligned}$$

From (75) and the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned} |\mathcal{R}(W^\tau)| \leq & \|V_0^1\|_{\mathcal{H}_2} \|W(0)\|_{\mathcal{H}_2} + C_T \|W(t)\|_{\mathcal{H}_2} \|(f_1, f_2, f_3)\|_{[L^2(0,T)]^3} \\ & + C_T \|W(t)\|_{C([0,\tau]; \mathcal{H}_2)} \|(J_1, J_2, J_3)\|_{(L^2(0,T; (H_*^1(0,L))^*)^3)}^3 \\ & + C_T \|W(t)\|_{C([0,\tau]; \mathcal{H}_2)} \|(u_t(L, 0), v_t(L, 0), w_t(L, 0))\|_{\mathbb{R}^3} \\ \leq & C_T \left(\|V_0^1\|_{\mathcal{H}_2} + \|(f_1, f_2, f_3)\|_{[L^2(0,T)]^3} + \|(J_1, J_2, J_3)\|_{(L^2(0,T; (H_*^1(0,L))^*)^3)}^3 \right. \\ & \left. + \|(u_t(L, 0), v_t(L, 0), w_t(L, 0))\|_{\mathbb{R}^3} \right) \|W^\tau\|_{\mathcal{H}_2}. \end{aligned}$$

Therefore, we have $\mathcal{R} \in \mathcal{L}(D(\mathcal{P}); \mathbb{R})$. By the Riesz representation theorem, it follows that there exists a unique $U \in D(\mathcal{P})$ satisfying (73). Moreover, the property $U \in C([0, T]; D(\mathcal{P}))$ follows by an argument analogous to that in [40], and thus the details are omitted. $\square$

5. Controllability results

In this section, we study some boundary controllability properties of system (8). We start with the following characterization of a control driving system (8) to zero. This kind of result is already classic for dispersive systems (see, for instance, [41] and [42]).

Lemma 5.1. *The initial data $U_0 \in D(\mathcal{P})$ is controllable to zero in time $T > 0$ with controls $f_i \in L^2(0, T)$, for $i = 1, 2, 3$, if and only if*

$$-\left\langle \begin{pmatrix} \rho_1 h_1 u_1 \\ -\rho_1 h_1 u_0 \\ \rho_3 h_3 v_1 \\ -\rho_3 h_3 v_0 \\ \rho h w_1 \\ -\rho h w_0 \\ -E_1 h_1 u(L, 0) \\ -E_3 h_3 v(L, 0) \\ -\alpha k w(L, 0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi_t(L, 0) \\ \eta_t(L, 0) \\ \Theta_t(L, 0) \end{pmatrix} \right\rangle_{\mathcal{H}_2} \quad (76)$$

$$\begin{aligned}
& - ((E_1 h_1 w_t(L, 0), E_3 h_3 v_t(L, 0), k \alpha w_t(L, 0)) ; (\chi(L, 0), \eta(L, 0), \Theta(L, 0)))_{\mathbb{R}^3} \\
& = E_1 h_1 \int_0^T f_1(t) \chi(L, t) dt + E_3 h_3 \int_0^T f_2(t) \eta(L, t) dt + k \alpha \int_0^T f_3(t) \Theta(L, t) dt
\end{aligned}$$

for any solution of the adjoint system

$$\begin{cases}
\rho_1 h_1 \chi_{tt} - E_1 h_1 \chi_{xx} - k(-\chi + \eta + \alpha \Theta_x) = 0, & x \in (0, L), \quad t > 0, \\
\rho_3 h_3 \eta_{tt} - E_3 h_3 \eta_{xx} + k(-\chi + \eta + \alpha \Theta_x) = 0, & x \in (0, L), \quad t > 0, \\
\rho h \Theta_{tt} - EI \Theta_{xxxx} - \alpha k(-\chi + \eta + \alpha \Theta_x)_x = 0, & x \in (0, L), \quad t > 0, \\
\chi(0, t) = \eta(0, t) = \Theta_x(0, t) = \Theta_{xxx}(0, t) = 0, & t > 0, \\
\Theta_{xx}(L, t) = \Theta_{xxx}(L, t) = 0, & t > 0, \\
\chi_t(L, t) + \chi_x(L, t) = 0, & t > 0, \\
\eta_t(L, t) + \eta_x(L, t) = 0, & t > 0, \\
\Theta_{tt}(L, t) - \chi_x(L, t) + \eta(L, t) + \alpha \Theta(L, t) = 0, & t > 0, \\
(\chi, \eta, \Theta)(x, T) = (\chi_0^T, \eta_0^T, \Theta_0^T)(x), & x \in (0, L), \\
(\chi_t, \eta_t, \Theta_t)(x, T) = (\chi_1^T, \eta_1^T, \Theta_1^T)(x), & x \in (0, L),
\end{cases} \quad (77)$$

with $(\chi_0^T, \chi_1^T, \eta_0^T, \eta_1^T, \Theta_0^T, \Theta_1^T, \chi_1^T(L), \eta_1^T(L), \Theta_1^T(L))^T \in D(\mathcal{P}^*)$.

Proof. The proof is a direct consequence of integration by parts, and so we will omit it. $\square$

Remark 5.1. It is well-known that the variational equation (76) has a solution if there exists a positive constant $C > 0$, such that the following observability inequality holds for any

$$\begin{aligned}
& (\chi_0^T, \chi_1^T, \eta_0^T, \eta_1^T, \Theta_0^T, \Theta_1^T, \chi_1^T(L), \eta_1^T(L), \Theta_1^T(L))^T \in D(\mathcal{P}^*), \\
& \|W(0)\|_{\mathcal{H}_2}^2 \leq C \int_0^T (|\chi(L, t)|^2 dt + |\eta(L, t)|^2 + |\Theta(L, t)|^2) dt,
\end{aligned} \quad (78)$$

where $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^T$ is the solution of the adjoint system (77).

Now, we are in a position to prove the observability inequality (78). We start with the following result.

Proposition 5.1. Let $U = (u, u_t, v, v_t, w, w_t, u_t(L, t), v_t(L, t), w_t(L, t))^T$ be the solution of system (51). Then, for any $T > 0$ and $s \in (\frac{1}{2}, 1)$, there exists a positive constant C such that the following estimate hold

$$\|U(t)\|_{\mathcal{H}_2}^2 \leq C \int_0^T (|u_t(L, t)|^2 dt + |v_t(L, t)|^2 + |w_t(L, t)|^2) dt + C \|(u, v, w)\|_{L^\infty(0, T; [H^s(0, L)]^3)}^2. \quad (79)$$

Proof. In order to obtain estimate (79), we multiply the first equation in (50) by xu_x and integrate by parts on $(0, L) \times (0, T)$ to obtain

$$\begin{aligned}
0 & = \rho_1 h_1 \int_0^L [xu_t u_x]_0^T dx - \frac{\rho_1 h_1 L}{2} \int_0^T u_t^2(L, t) dt + \frac{\rho_1 h_1}{2} \int_0^L \int_0^T u_t^2 dt dx \\
& \quad - \frac{E_1 h_1 L}{2} \int_0^T u_x^2(L, t) dt + \frac{E_1 h_1}{2} \int_0^L \int_0^T u_x^2 dt dx - k \int_0^L \int_0^T (-u + v + \alpha w_x) xu_x dt dx.
\end{aligned} \quad (80)$$

Now, we multiply the second equation in (50) by xv_x and the third one by xw_x , and integrate by parts on $(0, L) \times (0, T)$, to get

$$\begin{aligned} 0 &= \int_0^L \int_0^T (\rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x)) x v_x dt dx \\ &= \rho_3 h_3 \int_0^L [x v_t v_x]_0^T dx - \frac{\rho_3 h_3 L}{2} \int_0^T v_t^2(L, t) dt + \frac{\rho_3 h_3}{2} \int_0^L \int_0^T v_t^2 dt dx \\ &\quad - \frac{E_3 h_3 L}{2} \int_0^T v_x^2(L, t) dt + \frac{E_3 h_3}{2} \int_0^L \int_0^T v_x^2 dt dx - k \int_0^L \int_0^T (-u + v + \alpha w_x) x v_x dt dx \end{aligned} \quad (81)$$

and

$$\begin{aligned} 0 &= \int_0^L \int_0^T (\rho h w_{tt} + E I w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x) x w_x dt dx \\ &= \rho h \int_0^L [x w_t w_x]_0^T dx - \frac{\rho h L}{2} \int_0^T w_t^2(L, t) dt + \frac{\rho h}{2} \int_0^L \int_0^T w_t^2 dt dx \\ &\quad + \frac{3 E I}{2} \int_0^L \int_0^T w_{xx}^2 dt dx - k \alpha \int_0^T L(-u(L, t) + v(L, t) + \alpha w_x(L, t)) w_x(L, t) dt \\ &\quad + k \alpha \int_0^L \int_0^T (-u + v + \alpha w_x) x w_{xx} dt dx + k \int_0^L \int_0^T (-u + v + \alpha w_x)^2 dt dx \\ &\quad - k \int_0^L \int_0^T (-u + v + \alpha w_x)(-u + v) dt dx, \end{aligned} \quad (82)$$

respectively. A straightforward computation shows that

$$\begin{aligned} &k \int_0^L \int_0^T (-u + v + \alpha w_x)(-u + v + \alpha w_x)_x x dt dx \\ &= \frac{k L}{2} \int_0^T (-u(L, t) + v(L, t) + \alpha w_x(L, t))^2 dt - \frac{k}{2} \int_0^L \int_0^T (-u + v + \alpha w_x)^2 dt dx. \end{aligned} \quad (83)$$

Thus, adding the identities (80)–(82) and taking into account (83), it follows that

$$\begin{aligned} &\frac{1}{2} \int_0^T \|U(t)\|_{\mathcal{H}_2}^2 dt \\ &= \frac{1}{2} \int_0^T ((\rho_1 h_1 L + E_1 h_1) u_t^2(L, t) + ((\rho_3 h_3 L + E_3 h_3) v_t^2(L, t) + (\rho h L + \alpha k) w_t^2(L, t)) dt \\ &\quad - \int_0^L [\rho_1 h_1 x u_t u_x + \rho_3 h_3 x v_t v_x + \rho h x w_t w_x]_0^T dx + \frac{L}{2} \int_0^T (E_1 h_1 u_x^2(L, t) + E_3 h_3 v_x^2(L, t)) dt \\ &\quad + k \int_0^L \int_0^T (-u + v + \alpha w_x)(-u + v) dt dx \\ &\quad + \frac{\alpha k L}{2} \int_0^T (-u(L, t) + v(L, t) + \alpha w_x(L, t)) w_x(L, t) dt \\ &\quad - \frac{k L}{2} \int_0^T (-u(L, t) + v(L, t) + \alpha w_x(L, t))(-u(L, t) + v(L, t)) dt. \end{aligned} \quad (84)$$

Applying Young's inequality, for any $s \geq 0$ and $\varepsilon > 0$, we have that

$$\begin{aligned} & 3k \int_0^L (-u + v + \alpha w_x) (-u + v) dx \\ & \leq \varepsilon \left(k \|(-u + v + \alpha w_x)(\cdot, t)\|^2 + E_1 h_1 \|u_x(\cdot, t)\|^2 + E_3 h_3 \|v_x(\cdot, t)\|^2 \right) \\ & + C_{\varepsilon, k} \|(u, v)(\cdot, t)\|_{[H^s(0, L)]^2}^2. \end{aligned} \quad (85)$$

On the other hand, for $s \in (\frac{1}{2}, 1)$, using Young's inequality and Sobolev embedding (see, for instance, [43, 44]), we obtain a positive constant $C_{\varepsilon_1, k, E, L}$, such that

$$\begin{aligned} & \frac{L}{2} \left| \left(E_1 h_1 u_x^2(L, t) + E_3 h_3 v_x^2(L, t) + \alpha k (-u(L, t) + v(L, t) + \alpha w_x(L, t)) w_x(L, t) \right. \right. \\ & \quad \left. \left. - k (-u(L, t) + v(L, t) + \alpha w_x(L, t)) (-u(L, t) + v(L, t)) \right| \right. \\ & \leq \varepsilon_1 \left(k \|(-u + v + \alpha w_x)(\cdot, t)\|^2 + E_1 h_1 \|u_x(\cdot, t)\|^2 + E_3 h_3 \|v_x(\cdot, t)\|^2 \right) \\ & + C_{\varepsilon_1, k, E, L} \|(u, v, w)(\cdot, t)\|_{[H^s(0, L)]^3}^2, \end{aligned} \quad (86)$$

for any $\varepsilon_1 > 0$. Now, for all $t \in [0, T]$ and applying Young's inequality once more, we get that

$$\begin{aligned} & \left| \int_0^L (\rho_1 h_1 x u_t u_x + \rho_3 h_3 x v_t v_x + \rho h x w_t w_x)(x, t) dx \right| \\ & \leq \varepsilon_2 \left(\rho_1 h_1 \|u_t(\cdot, t)\|^2 + \rho_3 h_3 \|v_t(\cdot, t)\|^2 + \rho h \|w_t(\cdot, t)\|^2 \right) + C_{\varepsilon_2, \rho, L} \|(u, v, w)(\cdot, t)\|_{[H^s(0, L)]^3}^2, \end{aligned} \quad (87)$$

for any $\varepsilon_2 > 0$, $s \in (\frac{1}{2}, 1)$, and some constant $C_{\varepsilon_2, \rho_1, \rho_2, L} > 0$. By combining (84)–(87) and taking $\varepsilon = \varepsilon_1 = \varepsilon_2$, it follows that

$$\begin{aligned} \frac{1}{2} \int_0^T \|U(t)\|_{\mathcal{H}_2}^2 dt & \leq 2\varepsilon \int_0^T \|U(t)\|_{\mathcal{H}_2}^2 dt + C \int_0^T \left(u_t^2(L, t) + v_t^2(L, t) + w_t^2(L, t) \right) dt \\ & + C_\varepsilon \int_0^T \|(u, v, w)(\cdot, t)\|_{[H^s(0, L)]^3}^2 dt. \end{aligned}$$

From the above estimates, one can choose ε small enough so that $0 < \varepsilon < \frac{1}{4}$, and we immediately deduce

$$\int_0^T \|U(t)\|_{\mathcal{H}_2}^2 dt \leq C \int_0^T \left(u_t^2(L, t) + v_t^2(L, t) + w_t^2(L, t) \right) dt + C \|(u, v, w)\|_{L^\infty(0, T; [H^s(0, L)]^3)}^2, \quad (88)$$

for any $s \in (\frac{1}{2}, 1)$. Finally, using (68) and inequality (88), we obtain (80). This finishes the proof. $\square$

With the previous results in hand, we can establish an intermediary observability inequality as follows.

Proposition 5.2. *Let $U = (u, u_t, v, v_t, w, w_t, u_t(L, t), v_t(L, t), w_t(L, t))^T$ be the solution of system (51). Then, there exists a positive constant C such that the following estimate holds*

$$\|U_0\|_{\mathcal{H}_2}^2 \leq C \int_0^T \left(|u_t(L, t)|^2 + |v_t(L, t)|^2 + |w_t(L, t)|^2 \right) dt. \quad (89)$$

Proof. Let us argue by contradiction, following the so-called *compactness-uniqueness* argument (see, for instance, [45]). Suppose that (89) does not hold. Hence, there exists a sequence $U_0^n \in \mathcal{H}_2$ such that

$$\|U^n(t)\|_{\mathcal{H}_2}^2 = \|U_0^n\|_{\mathcal{H}_2}^2 = 1, \quad \text{for all } t \in [0, T], \quad (90)$$

and

$$\int_0^T \left(|u_t^n(L, t)|^2 + |v_t^n(L, t)|^2 + |w_t^n(L, t)|^2 \right) dt \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (91)$$

Here, for each $n \in \mathbb{N}$, the function

$$U^n = (u^n, u_t^n, v^n, v_t^n, w^n, w_t^n, u_t^n(L, t), v_t^n(L, t), w_t^n(L, t))^T$$

solves the following system:

$$\begin{cases} U_t^n(t) = \mathcal{P}U^n(t), \\ U^n(0) = U_0^n. \end{cases} \quad (92)$$

From (90) and the well-posedness results, we obtain the following uniform bounds:

$$\begin{cases} (u^n, v^n, w^n) & \text{bounded in } L^\infty(0, T; [H^1(0, L)]^3), \\ (u_t^n, v_t^n, w_t^n) & \text{bounded in } L^\infty(0, T; [L^2(0, L)]^3). \end{cases} \quad (93)$$

Once we have the embedding

$$H^1(0, L) \hookrightarrow H^s(0, L) \hookrightarrow L^2(0, L),$$

and since the embedding $H^1(0, L) \hookrightarrow H^s(0, L)$ is compact for all $s \in (\frac{1}{2}, 1)$, from (93) and the classical compactness results given by [46, Corollary 4], we can extract a subsequence of (u^n, v^n, w^n) , still denoted by (u^n, v^n, w^n) , such that

$$(u^n, v^n, w^n) \longrightarrow (u, v, w) \quad \text{strongly in } L^\infty(0, T; [H^s(0, L)]^3). \quad (94)$$

Thus, from (79), (91), and (94), we deduce that (U^n) is a Cauchy sequence in $L^\infty(0, T; \mathcal{H}_2)$. Therefore,

$$U^n \longrightarrow U \quad \text{strongly in } L^\infty(0, T; \mathcal{H}_2),$$

and by (90),

$$\|U(t)\|_{\mathcal{H}_2}^2 = 1, \quad \text{for all } t \in [0, T]. \quad (95)$$

Also,

$$\begin{aligned} 0 &= \liminf_{n \rightarrow \infty} \left\{ \int_0^T \left(|u_t^n(L, t)|^2 + |v_t^n(L, t)|^2 + |w_t^n(L, t)|^2 \right) dt \right\} \\ &\geq \int_0^T \left(|u_t(L, t)|^2 + |v_t(L, t)|^2 + |w_t(L, t)|^2 \right) dt. \end{aligned} \quad (96)$$

From (96), we have that the limit

$$U = (u, u_t, v, v_t, w, w_t, u_t(L, t), v_t(L, t), w_t(L, t))^T$$

satisfies the linear problem

$$\begin{cases} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, & x \in (0, L), t \in (0, T), \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, & x \in (0, L), t \in (0, T), \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, & x \in (0, L), t \in (0, T), \\ u(0, t) = v(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0, & t \in (0, T), \\ w_{xx}(L, t) = w_{xxx}(L, t) = 0, & t \in (0, T), \\ u_x(L, t) = 0, \quad v_x(L, t) = 0, & t \in (0, T), \\ -u(L, t) + v(L, t) + \alpha w_x(L, t) = 0, & t \in (0, T). \end{cases} \quad (97)$$

Let $(\hat{u}, \hat{v}, \hat{w})$ denote the extension by zero of (u_{tt}, v_{tt}, w_{tt}) for $x \in (-a, a) \setminus (0, L)$, where $(-a, a) \supset (0, L)$ is an interval. Then, $(\hat{u}, \hat{v}, \hat{w})$ solves, in $\mathcal{D}'((0, L) \times (0, T))$, the following system:

$$\begin{cases} \rho_1 h_1 \hat{u}_{tt} - E_1 h_1 \hat{u}_{xx} - k(-\hat{u} + \hat{v} + \alpha \hat{w}_x) = 0, & x \in (0, L), t \in (0, T), \\ \rho_3 h_3 \hat{v}_{tt} - E_3 h_3 \hat{v}_{xx} + k(-\hat{u} + \hat{v} + \alpha \hat{w}_x) = 0, & x \in (0, L), t \in (0, T), \\ \rho h \hat{w}_{tt} + EI \hat{w}_{xxxx} - \alpha k(-\hat{u} + \hat{v} + \alpha \hat{w}_x)_x = 0, & x \in (0, L), t \in (0, T), \\ \hat{u}(0, t) = \hat{v}(0, t) = \hat{w}_x(0, t) = \hat{w}_{xxx}(0, t) = 0, & t \in (0, T), \\ \hat{w}_{xx}(L, t) = \hat{w}_{xxx}(L, t) = 0, & t \in (0, T), \\ \hat{u}_x(L, t) = 0, & t \in (0, T), \\ \hat{v}_x(L, t) = 0, & t \in (0, T), \\ -\hat{u}(L, t) + \hat{v}(L, t) + \alpha \hat{w}_x(L, t) = 0, & t \in (0, T), \\ (\hat{u}, \hat{v}, \hat{w}) = (0, 0, 0), & x \in (-a, 0] \cup [L, a), t \in (0, T). \end{cases} \quad (98)$$

Then, by Holmgren's Uniqueness Theorem [47], $(\hat{u}, \hat{v}, \hat{w}) \equiv (0, 0, 0)$ in $(-a, a) \times (0, T)$. Hence, from (97), we have that $U = (u, u_t, v, v_t, w, w_t, u_t(L, t), v_t(L, t), w_t(L, t))^T$ satisfies

$$\begin{cases} -E_1 h_1 u_{xx} - k(-u + v + \alpha w_x) = 0, & x \in (0, L), t \in (0, T), \\ -E_3 h_3 v_{xx} + k(-u + v + \alpha w_x) = 0, & x \in (0, L), t \in (0, T), \\ EI w_{xxxx} - \alpha k(-u + v + \alpha w_x)_x = 0, & x \in (0, L), t \in (0, T), \\ u(0, t) = v(0, t) = w_x(0, t) = w_{xxx}(0, t) = 0, & t \in (0, T), \\ w_{xx}(L, t) = w_{xxx}(L, t) = 0, & t \in (0, T), \\ u_x(L, t) = 0, & t \in (0, T), \\ v_x(L, t) = 0, & t \in (0, T), \\ -u(L, t) + v(L, t) + \alpha w_x(L, t) = 0, & t \in (0, T). \end{cases} \quad (99)$$

We infer, from the proof of Lemma 4.1, the unique solution of (99) is given by $U = 0$, and this contradicts (95). Then, necessarily, (89) holds. This completes the proof of the theorem. $\square$

Remark 5.2. From (68), we directly verify that

$$\int_0^T (|u_t(L, t)|^2 + |v_t(L, t)|^2 + |w_t(L, t)|^2) dt \leq C \|U_0\|_{\mathcal{H}_2}^2. \quad (100)$$

We are now in a position to prove the second main result of this section, namely, the observability inequality that provides a positive answer to the control problem P_3 .

Theorem 5.1. *For any $U_0 \in D(\mathcal{P})$, let*

$$U = (u, u_t, v, v_t, w, w_t, u_t(L, t), v_t(L, t), w_t(L, t))^T$$

be the solution of system (51). Then, there exists a positive constant C such that the following estimate holds:

$$\|U_0\|_{\mathcal{H}_2}^2 \leq C \int_0^T (|u(L, t)|^2 + |v(L, t)|^2 + |w(L, t)|^2) dt. \quad (101)$$

Proof. In order to prove inequality (101), we consider the auxiliary problem

$$\begin{cases} \tilde{U}_t = \mathcal{P}\tilde{U}, \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \quad (102)$$

where $\tilde{U}_0 = \mathcal{P}^{-1}U_0$ (the existence of $\mathcal{P}^{-1}$ is guaranteed by Lemma 4.1). Thus, from Proposition 5.2, the solution

$$\tilde{U} = (\tilde{u}, \tilde{u}_t, \tilde{v}, \tilde{v}_t, \tilde{w}, \tilde{w}_t, \tilde{u}_t(L, t), \tilde{v}_t(L, t), \tilde{w}_t(L, t))^T$$

of system (102) satisfies

$$\|\tilde{U}_0\|_{\mathcal{H}_2}^2 \leq C \int_0^T (|\tilde{u}_t(L, t)|^2 + |\tilde{v}_t(L, t)|^2 + |\tilde{w}_t(L, t)|^2) dt, \quad (103)$$

for some constant $C > 0$.

On the other hand, it is straightforward to see that $\tilde{U}_t = U$. Hence,

$$(\tilde{u}_t, \tilde{v}_t, \tilde{w}_t) = (u, v, w). \quad (104)$$

Finally, combining (103) with (104), and applying the Riesz representation theorem, we deduce that

$$\begin{aligned} \|U_0\|_{\mathcal{H}_2}^2 &\leq C \|\tilde{U}_0\|_{\mathcal{H}_2}^2 \\ &\leq C \int_0^T (|\tilde{u}_t(L, t)|^2 + |\tilde{v}_t(L, t)|^2 + |\tilde{w}_t(L, t)|^2) dt \\ &= C \int_0^T (|u(L, t)|^2 + |v(L, t)|^2 + |w(L, t)|^2) dt, \end{aligned}$$

and the proof is complete. $\square$

A direct consequence of the previous theorem is the following.

Corollary 5.1. *Let*

$$W = (\chi, \chi_t, \Theta, \Theta_t, \eta, \eta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^T$$

be the solution of the adjoint system (77). Then, there exists a constant $C > 0$ such that the following observability inequality holds:

$$\|W(0)\|_{\mathcal{H}_2}^2 \leq C \int_0^T (|\chi(L, t)|^2 + |\eta(L, t)|^2 + |\Theta(L, t)|^2) dt. \quad (105)$$

The observability inequality established in Corollary 5.1, combined with the HUM method from control theory, allows us to obtain the second main outcome of this paper, namely, the controllability of the system stated in Theorem 1.2.

6. Conclusion

This paper considered two different problems related to a linear Rao-Nakra-type sandwich beam, which consists of three coupled PDEs. An exponential stabilization result is obtained for the first problem, where the boundary conditions are static and subject to three time-dependent boundary delayed controls, while three time-varying interior damping controls are present. It would be desirable to investigate, in future work, the stabilization problem with the presence of at most two interior damping controls. Furthermore, a promising research avenue is to tackle the problem where the boundary controls are of memory-type.

For the second problem, the boundary conditions are supposed to be dynamical. Then, a null controllability result is shown via the action of three boundary controls. Another interesting question is whether one can achieve the controllability result by means of at most two boundary controls.

Author contributions

All authors contributed equally to the conceptualization, methodology, formal analysis, investigation, original draft preparation, review and editing of the manuscript, and have approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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