

ORIGINAL PAPER

On the Controllability of Laminated Beams With Venttsel-Type Boundary Conditions

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ABSTRACT

This paper examines the boundary controllability of a Timoshenko laminated beam system subject to Venttsel-type boundary conditions. The study focuses on a novel configuration in which three controls are applied solely at the boundary of the beam. Controllability is established by deriving an appropriate observability inequality for the corresponding adjoint system, which is then employed within the framework of the duality method in the setup of the classical Hilbert uniqueness method (HUM) to achieve the control problem. The main contribution lies in the analysis of a system comprising three beams governed by dynamic Venttsel-type boundary conditions, as introduced by Venttsel (1959).

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1 | Introduction

1.1 | Model Under Consideration

This paper focuses on the analysis of the solution behavior for a Timoshenko-type laminated beam model governed by the following system

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, & \text{in } (0, L) \times \mathbb{R}^+, \\ I_\rho(3S_{tt} - \psi_{tt}) - D(3S_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0, L) \times \mathbb{R}^+, \\ 3I_\rho S_{tt} - 3DS_{xx} + 3G(\psi - w_x) + 4\delta_0 S + 4\gamma_0 S_t = 0, & \text{in } (0, L) \times \mathbb{R}^+, \end{cases} \quad (1.1)$$

subject to homogeneous dynamic Venttsel-type boundary conditions and prescribed initial data [1]. Here, L denotes the length of the beam, and the subscripts x and t indicate partial derivatives with respect to the spatial and temporal variables, respectively. In Equation (1.1), $w(x, t)$ denotes the transverse displacement, and $\psi(x, t)$ represents the angle of rotation. The function $S(x, t)$ corresponds to the relative slip along the adhesive interface at position x and time t . The parameters

ρ , G , I_ρ , D , δ_0 , and γ_0 are all positive constants, representing the density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive structural damping coefficient, respectively.

If we change the variables $s = -3S$, $\xi = 3S - \psi$, $\rho_1 = \rho$, $\rho_2 = I_\rho$, $k = G$, $b = D$, $3\gamma = 4\delta_0$, $3\beta = 4\gamma_0$, then the system (1.1) becomes

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = 0, & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = 0, & \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s + \beta s_t = 0, & \text{in } (0, L) \times \mathbb{R}^+. \end{cases} \quad (1.2)$$

Laminated beam models (1.2) have attracted significant interest due to the growing use of laminated materials in engineering applications, particularly in the automotive, aerospace, and construction industries, where their high strength, durability, and flexibility are highly valued (see, e.g., Mahajan [2] and Reddy [3]). The model considered here, originally developed by Hansen and Spies [4] from Timoshenko's classical beam theory [5], describes a two-layer beam joined by a thin adhesive layer of negligible mass and thickness. This layer contributes a restoring force proportional to the interfacial slip. While the first two equations in (1.2) stem from Timoshenko theory, the third captures the interlayer interaction and incorporates structural damping through an internal friction term.

So, in context, we are interested in the control properties of the laminated beam models (1.2) subject to a class of homogeneous dynamic Venttsel-type¹ boundary conditions and prescribed initial data, namely

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s = 0, & x \in (0, L), \quad t > 0, \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t > 0, \\ w_{tt}(L, t) + w_x(L, t) + \xi(L, t) + s(L, t) = u_1(t), & t > 0, \\ \xi_{tt}(L, t) + \xi_x(L, t) = u_2(t), & t > 0, \\ s_{tt}(L, t) + s_x(L, t) = u_3(t), & t > 0, \\ (w, \xi, s)(x, 0) = (w_0, \xi_0, s_0)(x), & x \in (0, L), \\ (w_t, \xi_t, s_t)(x, 0) = (w_1, \xi_1, s_1)(x), & x \in (0, L). \end{cases} \quad (1.3)$$

To be precise, this work aims to determine whether suitable boundary controls $u_1(t)$, $u_2(t)$, and $u_3(t)$ applied at the boundary of the beam can steer the system's solutions to exhibit prescribed behaviors. This inquiry touches on a central problem in control theory:

Null Controllability Problem: *Given $T > 0$ and initial states $(w_0, w_1, \xi_0, \xi_1, s_0, s_1)$ within a specified function space, can we determine a control inputs u_1 , u_2 , and u_3 such that the system (1.3) admits a solution (w, ξ, s) satisfying the initial conditions and the null control properties*

$$(w, \xi, s)(x, T) = (0, 0, 0) \quad \text{and} \quad (w_t, \xi_t, s_t)(x, T) = (0, 0, 0), \quad \text{for } x \in (0, L)?$$

1.2 | Background

Venttsel-type boundary conditions emerged independently in the works of Venttsel [6] and Feller [7], originally in the study of generalized second-order differential operators and their connections to Markov processes. A broader framework was later developed in [1]. In recent years, there has been growing interest in evolution equations incorporating Venttsel and dynamic boundary conditions, driven by their relevance in several applied contexts.

It is essential to note that these types of conditions occur in various mathematical models, particularly in fields such as population dynamics, heat transfer, continuum thermodynamics, and planar motion. For additional context and examples, the reader may consult references [8–13] and the literature cited therein. A comprehensive treatment of the physical background and derivation of these boundary conditions can be found in the seminal work [14].

These boundary conditions have been the focus of significant research in the context of control and stabilization problems for partial differential equations. Several works address the stabilization of the wave equation under such boundary

conditions. For instance, the exponential decay of energy has been established in domains of $\mathbb{R}^n$ and on compact Riemannian manifolds, as discussed in [15–19], among others.

In the context of parabolic equations, the boundary null controllability of the heat equation with Wentzell-type boundary conditions has been studied in [20]. These conditions are particularly relevant in modeling physical systems where the boundary exhibits its dynamics, such as thermoelastic or fluid–structure interactions.

Boundary stabilization under dynamical boundary conditions has also been considered for various models. In [15–17], the authors analyze the Cauchy–Ventcel problem, establishing uniform energy decay rates. In [16], stabilization and observability estimates, along with asymptotic energy decay rates, are established for the wave equation with nonlinear interior damping and Wentzell-type boundary conditions. Also, we mention that in [17], the authors investigate a wave equation subject to nonlinear interior damping and Wentzell-type boundary conditions of the form

$$\partial_\nu u + u = \Delta_T u.$$

They establish stabilization and observability estimates, as well as asymptotic energy decay rates for the system. A notable aspect of this work is that the damping mechanism is localized in a subset of the interior and does not extend to a full collar of the boundary, leaving a portion governed exclusively by the high-order Wentzell condition. This partial dissipation poses additional challenges in analyzing the long-time behavior of the solution. More recently, in [15], he studies the damped wave equation on either an open subset of $\mathbb{R}^n$ or a smooth Riemannian manifold with boundary, under Venttsel-type boundary conditions. By employing refined Carleman estimates near the boundary, they derive resolvent estimates along the imaginary axis, which yield logarithmic decay rates for the associated energy, uniform with respect to the thickness parameter δ .

Further investigations include the analysis of coupled systems with dynamical boundary interactions. For example, in [21], the authors examine a one-dimensional Timoshenko beam system. In the first part of their work, they consider a single boundary fractional damping and prove strong stability without uniformity. They then establish polynomial energy decay rates depending on the system coefficients and the order of the damping. In the second part, the study focuses on exact boundary controllability under mixed Dirichlet–Neumann conditions. Using non-harmonic Fourier analysis and the Hilbert uniqueness method (HUM), they derive controllability results in appropriate functional spaces.

Another notable contribution is found in [22], where the authors investigate a Cauchy–Ventcel problem in an inhomogeneous medium, subject to nonlinear boundary damping distributed in a neighborhood ω of the boundary, satisfying the geometric control condition. They obtain results on energy decay and stabilization in this nonlinear and spatially heterogeneous setting.

Lastly, several works address the exponential stabilization of laminated beams with structural damping and boundary feedback controls. These studies consider mechanical models of layered composite materials and utilize boundary actuators to enforce rapid decay of vibrations, further illustrating the relevance of dynamical boundary conditions in practical control scenarios (see, e.g., [23–26] and the references therein).

We emphasize that this represents only a small portion of the existing literature, and we encourage interested readers to consult the references cited in the works mentioned above.

1.3 | Main Result and Outline of the Paper

With this overview of the literature, we are in a position to present our main result. Before presenting the main result, let us introduce some notations. Consider the operator $\mathcal{A}$ associated to (1.3) defined by

$$\mathcal{A}U := \begin{bmatrix} \Psi_1 \\ \frac{1}{\rho_1}[k(w_x + \xi + s)_x] \\ \Psi_2 \\ \frac{1}{\rho_2}[b\xi_{xx} - k(w_x + \xi + s)] \\ \Psi_3 \\ \frac{1}{\rho_2}[bs_{xx} - 3k(w_x + \xi + s) - \gamma s] \\ -(w_x(L) + \xi(L) + s(L)) \\ -\xi_x(L) \\ -s_x(L) \end{bmatrix}. \quad (1.4)$$

Here, the new variables are

$$\Psi_1 = w_t, \quad \Psi_2 = \xi_t, \quad \Psi_3 = s_t, \quad \Psi_4(\cdot) = \Psi_1(L, \cdot), \quad \Psi_5(\cdot) = \Psi_2(L, \cdot), \quad \Psi_6(\cdot) = \Psi_3(L, \cdot),$$

and define the vector functions

$$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \text{ and } U_0 = (w_0, w_1, \xi_0, \xi_1, s_0, s_1, \Psi_4(0), \Psi_5(0), \Psi_6(0))^\top.$$

Now, let us define the phase space $\mathcal{H}$, we consider the Lebesgue space $L^2(0, L)$ with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We will also use the following Hilbert space:

$$H_*^1(0, L) = \{w \in H^1(0, L); w(0) = 0\},$$

and define the phase space as

$$\mathcal{H} = [H_*^1(0, L) \times L^2(0, L)]^3 \times \mathbb{R}^3.$$

Endowed this space with the inner product

$$\begin{aligned} \langle U, U^\# \rangle_{\mathcal{H}} = & 3\rho_1 \langle \Psi_1, \Psi_1^\# \rangle + 3\rho_2 \langle \Psi_2, \Psi_2^\# \rangle + \rho_2 \langle \Psi_3, \Psi_3^\# \rangle + 3b \langle \xi_x, \xi_x^\# \rangle \\ & + b \langle s_x, s_x^\# \rangle + \gamma \langle s, s^\# \rangle + 3k \langle w_x + \xi + s, w_x^\# + \xi^\# + s^\# \rangle \\ & + 3k \left(\Psi_4, \Psi_4^\# \right)_{\mathbb{R}} + 3b \left(\Psi_5, \Psi_5^\# \right)_{\mathbb{R}} + b \left(\Psi_6, \Psi_6^\# \right)_{\mathbb{R}} \end{aligned}$$

and the following norm

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 = & 3\rho_1 \|\Psi_1\|^2 + 3\rho_2 \|\Psi_2\|^2 + \rho_2 \|\Psi_3\|^2 + 3b \|\xi_x\|^2 + b \|s_x\|^2 + \gamma \|s\|^2 + 3k \|w_x + \xi + s\|^2 \\ & + 3k |\Psi_4|_{\mathbb{R}}^2 + 3b |\Psi_5|_{\mathbb{R}}^2 + b |\Psi_6|_{\mathbb{R}}^2, \end{aligned}$$

for $U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top$, $U^\# = (w^\#, \Psi_1^\#, \xi^\#, \Psi_2^\#, s^\#, \Psi_3^\#, \Psi_4^\#, \Psi_5^\#, \Psi_6^\#)^\top \in \mathcal{H}$. So, the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ defined by (1.4) has domain given by

$$D(\mathcal{A}) = \left\{ \begin{aligned} & U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in \mathcal{H}; (\omega, \xi, s) \in [H^2(0, L) \cap H_*^1(0, L)]^3, \\ & (\Psi_1, \Psi_2, \Psi_3) \in [H_*^1(0, L)]^3, \Psi_4 = \Psi_1(L), \Psi_5 = \Psi_2(L), \Psi_6 = \Psi_3(L), \\ & (w_x(L), \xi_x(L), s_x(L)) \in \mathbb{R}^3. \end{aligned} \right\}$$

With this in hand, the following result ensures the controllability of the system (1.3) considering controls acting in the Venttsel-type boundary conditions.

Theorem 1.1. *Let a time $T > 0$ and given*

$$(w_0, w_1, \xi_0, \xi_1, s_0, s_1, w_1(L), \xi_1(L), s_1(L))^\top \in D(\mathcal{A}),$$

one can always find a control inputs $u_i \in L^2(0, T)$, $i = 1, 2, 3$, such that (1.3) admits a unique solution

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^\top \in C([0, T]; D(\mathcal{A}))$$

satisfying

$$(w(T), w_t(T), \xi(T), \xi_t(T), s(T), s_t(T), w_t(L, T), \xi_t(L, T), s_t(L, T)) = (0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Moreover, there exists a constant $C > 0$ such that

$$\|(u_1, u_2, u_3)\|_{[L^2(0,T)]^3} \leq C \|(w_0, w_1, \xi_0, \xi_1, s_0, s_1, w_1(L), \xi_1(L), s_1(L))^\top\|_H.$$

To establish Theorem 1.1, we employ the observation–control duality framework developed by Dolecki and Russell [27], within the setting introduced by Lions [28]. Specifically, we rely on the Hilbert uniqueness method (HUM), which provides a fundamental equivalence between controllability and observability. A key contribution of our work is to address a Timoshenko-type system comprising three coupled equations. This significantly increases the complexity of the analysis, as the model consists of three beams subject to dynamical boundary conditions. These boundary conditions introduce notable difficulties in establishing the observability inequality. To overcome this, we derive two intermediate inequalities—see (3.4) and (3.15) in Section 3—which play a crucial role in handling the full system.

Remark 1.1. We point out the following consequences of Theorem 1.1.

- a) Note that, due to the specific structure of the model, we expect controllability to hold for any time $T > 0$. Nevertheless, we believe that it may be possible to modify the boundary dynamics and apply the moment method (see, for instance, [29]), together with an Ingham-type theorem [30], to determine a minimal time for which exact controllability holds, or alternatively to identify a time threshold below which controllability fails.
- b) About the regularity and structure of the HUM controls:
 - (i) For initial data $U_0 \in H'$, the HUM method yields boundary controls $u_1, u_2, u_3 \in L^2(0, T)$, which are consistent with the observability inequality established in the paper.
 - (ii) If the initial data belong to smoother spaces, such as $\mathcal{H}$ or $D(A)$, then the solutions of the adjoint system possess higher regularity. As a consequence, the traces $\chi(L, t)$, $\eta(L, t)$, and $\Theta(L, t)$ are more regular, and the corresponding HUM controls inherit this improved regularity (for instance, belonging to $H^1(0, T)$).
 - (iii) Due to the dynamic Venttsel-type boundary conditions, the trace variables at $x = L$ are incorporated into the phase space. Therefore, no additional regularity assumptions on the boundary traces are required to ensure well-posedness of the forward controlled system.

We conclude the introduction with an outline of the paper. In Section 2, we prove the well-posedness of system (1.3) using semigroup theory. Section 3 is devoted to the proof of our main result. We begin by establishing an observability inequality for the adjoint system associated with (1.3), which leads to the proof of Theorem 1.1. Finally, in Section 4, we present concluding remarks and discuss directions for future research.

2 | Well-posedness Theory

This section is dedicated to establishing the well-posedness of system (1.3). We begin by analyzing the homogeneous case and then extend the results to the nonhomogeneous system. These foundational results are essential for addressing the controllability problem outlined in the introduction.

2.1 | The Homogeneous System

Let us first consider the following homogeneous system

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s = 0, & x \in (0, L), \quad t > 0, \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t > 0, \\ w_{tt}(L, t) + w_x(L, t) + \xi(L, t) + s(L, t) = 0, & t > 0, \\ \xi_{tt}(L, t) + \xi_x(L, t) = 0, & t > 0, \\ s_{tt}(L, t) + s_x(L, t) = 0, & t > 0, \\ (w, \xi, s)(x, 0) = (w_0, \xi_0, s_0)(x), & x \in (0, L), \\ (w_t, \xi_t, s_t)(x, 0) = (w_1, \xi_1, s_1)(x), & x \in (0, L). \end{cases} \quad (2.1)$$

Remember that

$$\Psi_1 = w_t, \quad \Psi_2 = \xi_t, \quad \Psi_3 = s_t, \quad \Psi_4(\cdot) = \Psi_1(L, \cdot), \quad \Psi_5(\cdot) = \Psi_2(L, \cdot), \quad \Psi_6(\cdot) = \Psi_3(L, \cdot),$$

and

$$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \text{ and } U_0 = (w_0, w_1, \xi_0, \xi_1, s_0, s_1, \Psi_4(0), \Psi_5(0), \Psi_6(0))^\top.$$

With these notations in hand, the system (2.1) can be written as an abstract evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \quad (2.2)$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U := \begin{bmatrix} \Psi_1 \\ \frac{1}{\rho_1}[k(w_x + \xi + s)_x] \\ \Psi_2 \\ \frac{1}{\rho_2}[b\xi_{xx} - k(w_x + \xi + s)] \\ \Psi_3 \\ \frac{1}{\rho_2}[bs_{xx} - 3k(w_x + \xi + s) - \gamma s] \\ -(w_x(L) + \xi(L) + s(L)) \\ -\xi_x(L) \\ -s_x(L) \end{bmatrix}. \quad (2.3)$$

Taking into account the spaces defined in the introduction and denoting by $\rho(\mathcal{A})$ the resolvent set of the operator $\mathcal{A}$. The following result ensures the invertibility of the operator $\mathcal{A}$.

Lemma 2.1. *Let $\mathcal{H}$ and $(\mathcal{A}, D(\mathcal{A}))$ be defined as before. Then, $0 \in \rho(\mathcal{A})$. Moreover, $\mathcal{A}^{-1}$ is compact.*

Proof. For $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) \in \mathcal{H}$, we will show the existence of

$$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in D(\mathcal{A}),$$

unique solution of the equation

$$\mathcal{A}U = F.$$

Equivalently, one must consider the system given by

$$\begin{cases} \Psi_1 = f_1, & \Psi_2 = f_3, & \Psi_3 = f_5, \\ k(w_x + \xi + s)_x = \rho_1 f_2, \\ b\xi_{xx} - k(w_x + \xi + s) = \rho_2 f_4, \\ bs_{xx} - 3k(w_x + \xi + s) - \gamma s = \rho_2 f_6, \\ -(w_x(L) + \xi(L) + s(L)) = f_7, \\ -\xi_x(L) = f_8, \\ -s_x(L) = f_9, \end{cases} \quad (2.4)$$

with the following boundary conditions

$$w(0) = \xi(0) = s(0) = 0. \quad (2.5)$$

Consider $(w^*, \xi^*, s^*) \in [H_*^1(0, L)]^3$, so multiplying the equation (2.4)₂, (2.4)₃, (2.4)₄, (2.4)₅, (2.4)₆ and (2.4)₇ by $3w^*$, $3\xi^*$, s^* , $3kw^*(L)$, $3b\xi^*(L)$, $bs^*(L)$, respectively, integrating by parts in $(0, L)$ and taking the sum, we get that

$$B((w, \xi, s), (w^*, \xi^*, s^*)) = \mathcal{L}(w^*, \xi^*, s^*), \quad \forall (w^*, \xi^*, s^*) \in [H_*^1(0, L)]^3, \quad (2.6)$$

thanks to the boundary conditions (2.5). Here

$$B((w, \xi, s), (w^*, \xi^*, s^*)) = 3b\langle \xi_x, \xi_x^* \rangle + b\langle s_x, s_x^* \rangle + \gamma\langle s, s^* \rangle + 3k\langle w_x + \xi + s, w_x^* + \xi^* + s^* \rangle, \quad (2.7)$$

and

$$\begin{aligned} \mathcal{L}(w^*, \xi^*, s^*) = & -3\rho_1\langle f_1, w^* \rangle - 3\rho_2\langle f_4, \xi^* \rangle - \rho_2\langle f_6, s^* \rangle \\ & + 3kf_7w^*(L) + 3bf_8\xi^*(L) + bf_9s^*(L). \end{aligned} \quad (2.8)$$

Now, observe that

$$B : [H_*^1(0, L)]^3 \times [H_*^1(0, L)]^3 \longrightarrow \mathbb{C}$$

defined by (2.7) is a bilinear form which is continuous and coercive. Moreover,

$$\mathcal{L} : [H_*^1(0, L)]^3 \longrightarrow \mathbb{C},$$

given by (2.8), is a linear continuous form. Then, using the Lax–Milgram theorem, we deduce that there exists $(w, \xi, s) \in [H_*^1(0, L)]^3$ a unique weak solution of the variational problem (2.6). From (2.4)₁ and applying the classical elliptic regularity theory we deduce that

$$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in D(\mathcal{A})$$

and, consequently, $\mathcal{A}$ is bijective.

Following similar calculations as done in [31, Proposition 2], we prove that $\mathcal{A}^{-1}$ is bounded. Thus, $0 \in \rho(\mathcal{A})$. Finally, by Sobolev's embedding theorem, we deduce that the canonical injection

$$i : D(\mathcal{A}) \hookrightarrow \mathcal{H}$$

is compact, and therefore, $\mathcal{A}^{-1}$ is also compact, showing the proposition. $\square$

The next result ensures that $\mathcal{A}$ generates a group. The result can be read as follows.

Theorem 2.1. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is an infinitesimal generator of a group of isometries $\{S(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$.*

Proof. Let

$$U^\sharp = (w^\sharp, \Psi_1^\sharp, \xi^\sharp, \Psi_2^\sharp, s^\sharp, \Psi_3^\sharp, \Psi_4^\sharp, \Psi_5^\sharp, \Psi_6^\sharp)^\top \in D(\mathcal{A}).$$

Then, for all

$$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in D(\mathcal{A}),$$

we deduce that

$$\langle \mathcal{A}U, U^\sharp \rangle_{\mathcal{H}} = 3\rho_1 \left\langle \frac{1}{\rho_1} [k(w_x + \xi + s)_x], \Psi_1^\sharp \right\rangle + 3\rho_2 \left\langle \frac{1}{\rho_2} [b\xi_{xx} - k(w_x + \xi + s)], \Psi_2^\sharp \right\rangle$$

$$\begin{aligned}
& + \rho_2 \left\langle \frac{1}{\rho_2} [bs_{xx} - 3k(w_x + \xi + s) - \gamma s], \Psi_3^\# \right\rangle + 3b \left\langle \Psi_{2,x}, \xi_x^\# \right\rangle + b \left\langle \Psi_{3,x}, s_x^\# \right\rangle \\
& + \gamma \left\langle \Psi_3, s^\# \right\rangle + 3k \left\langle \Psi_{1,x} + \Psi_2 + \Psi_3, w_x^\# + \xi^\# + s^\# \right\rangle \\
& + 3k \left(-(w_x(L) + \xi(L) + s(L)), \Psi_4^\# \right)_\mathbb{R} + 3b \left(-\xi_x(L), \Psi_5^\# \right)_\mathbb{R} + b \left(-s_x(L), \Psi_6^\# \right)_\mathbb{R} \\
& = 3\rho_1 \left\langle \Psi_1, -\frac{1}{\rho_1} [k(w_x^\# + \xi^\# + s^\#)_x] \right\rangle + 3\rho_2 \left\langle \Psi_2, -\frac{1}{\rho_2} [b\xi_{xx}^\# - k(w_x^\# + \xi^\# + s^\#)] \right\rangle \\
& + \rho_2 \left\langle \Psi_3, -\frac{1}{\rho_2} [bs_{xx}^\# - 3k(w_x^\# + \xi^\# + s^\#) - \gamma s^\#] \right\rangle + 3b \left\langle \xi_x, -\Psi_{2,x}^\# \right\rangle \\
& + b \left\langle s_x, -\Psi_{3,x}^\# \right\rangle + \gamma \left\langle s, -\Psi_3^\# \right\rangle + 3k \left\langle w_x + \xi + s, -(\Psi_{1,x}^\# + \Psi_2^\# + \Psi_3^\#) \right\rangle \\
& + 3k \left(\Psi_4, (w_x^\#(L) + \xi^\#(L) + s^\#(L)) \right)_\mathbb{R} \\
& + 3b \left(\Psi_5, \xi_x^\#(L) \right)_\mathbb{R} + b \left(\Psi_6, s_x^\#(L) \right)_\mathbb{R} \\
& = \left\langle U, -\mathcal{A}U^\# \right\rangle_{\mathcal{H}}.
\end{aligned}$$

Hence, $U^\# \in D(\mathcal{A}^*)$ and $\mathcal{A}^*U^\# = -\mathcal{A}U^\#$, for any $U^\# \in D(\mathcal{A})$.

On the other hand, let $U^\# = (w^\#, \Psi_1^\#, \xi^\#, \Psi_2^\#, s^\#, \Psi_3^\#, \Psi_4^\#, \Psi_5^\#, \Psi_6^\#)^\top \in D(\mathcal{A}^*)$. Then, there exists $U_0 \in \mathcal{H}$, such that

$$\left\langle \mathcal{A}U, U^\# \right\rangle_{\mathcal{H}} = \langle U, U_0 \rangle_{\mathcal{H}}, \text{ for all } U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in D(\mathcal{A}). \quad (2.9)$$

By the same calculations as before, we have that

$$\begin{aligned}
\left\langle \mathcal{A}U, U^\# \right\rangle_{\mathcal{H}} & = \left\langle U, -\mathcal{A}U^\# \right\rangle_{\mathcal{H}} + 3k \left((w_x(L) + \xi(L) + s(L)), \Psi_1^\#(L) - \Psi_4^\# \right)_\mathbb{R} \\
& + 3b \left(\xi_x(L), \Psi_2^\#(L) - \Psi_5^\# \right)_\mathbb{R} + b \left(s_x(L), \Psi_3^\#(L) - \Psi_6^\# \right)_\mathbb{R} \\
& - 3k \left(w_x(0), \Psi_1^\#(0) \right)_\mathbb{R} - 3b \left(\xi_x(0), \Psi_2^\#(0) \right)_\mathbb{R} - b \left(s_x(0), \Psi_3^\#(0) \right)_\mathbb{R}.
\end{aligned} \quad (2.10)$$

Therefore, taking the difference between (2.9) and (2.10), we obtain the following identity

$$\begin{aligned}
0 & = \left\langle U, U_0 + \mathcal{A}U^\# \right\rangle_{\mathcal{H}} + 3k \left((w_x(L) + \xi(L) + s(L)), \Psi_4^\# - \Psi_1^\#(L) \right)_\mathbb{R} \\
& + 3b \left(\xi_x(L), \Psi_5^\# - \Psi_2^\#(L) \right)_\mathbb{R} + b \left(s_x(L), \Psi_6^\# - \Psi_3^\#(L) \right)_\mathbb{R} + 3k \left(w_x(0), \Psi_1^\#(0) \right)_\mathbb{R} \\
& + 3b \left(\xi_x(0), \Psi_2^\#(0) \right)_\mathbb{R} + b \left(s_x(0), \Psi_3^\#(0) \right)_\mathbb{R},
\end{aligned}$$

for all $U \in D(\mathcal{A})$. From the above equality, we obtain that $U^\# \in D(\mathcal{A})$ and

$$\mathcal{A}^*U^\# = U_0 = -\mathcal{A}U^\#, \text{ for any } U^\# \in D(\mathcal{A}^*).$$

Thus, $D(\mathcal{A}) = D(\mathcal{A}^*)$ and $\mathcal{A}^* = -\mathcal{A}$. We deduce that $\mathcal{A}$ is a skew-adjoint operator and, therefore, generates a group of isometries $\{S(t)\}_{t \in \mathbb{R}}$ in $\mathcal{H}$, giving the result. $\square$

As a direct consequence of Lemma 2.1 and Theorem 2.1, and by applying semigroup theory for evolution equations², we obtain the following existence and uniqueness result.

Theorem 2.2. For any $U_0 \in D(\mathcal{A})$, there exists a unique solution U of the system (2.2) such that

$$U \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H}).$$

Moreover, for any $t > 0$, we have that

$$\|U(t)\|_{\mathcal{H}}^2 = \|U_0\|_{\mathcal{H}}^2. \quad (2.11)$$

2.2 | The Nonhomogeneous System

In this subsection, we focus on the complete system (1.3) and begin by presenting the following well-posedness result.

Theorem 2.3. For any $U_0 \in D(\mathcal{A})$ and $u_i \in C_0^\infty(0, \infty)$, for $i = 1, 2, 3$, system (1.3) has a unique solution $U \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H})$.

Proof. Let $\phi_1 \in C^\infty[0, L]$ be a cut-off function, such that $\phi_1(0) = \phi_1(L) = 0$ and $\phi_1'(L) = -1$. First, consider the change of variables

$$\begin{pmatrix} z \\ \eta \\ \varphi \end{pmatrix} = \begin{pmatrix} w \\ \xi \\ s \end{pmatrix} - \begin{pmatrix} \tilde{w} \\ \tilde{\xi} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} u_1(t)\phi_1(x) \\ u_2(t)\phi_1(x) \\ u_3(t)\phi_1(x) \end{pmatrix}. \quad (2.12)$$

Here, $(\tilde{w}, \tilde{\xi}, \tilde{s})$ is the unique solution of the system

$$\begin{cases} \rho_1 \tilde{w}_{tt} - k(\tilde{w}_x + \tilde{\xi} + \tilde{s})_x = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \tilde{\xi}_{tt} - b\tilde{\xi}_{xx} + k(\tilde{w}_x + \tilde{\xi} + \tilde{s}) = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \tilde{s}_{tt} - b\tilde{s}_{xx} + 3k(\tilde{w}_x + \tilde{\xi} + \tilde{s}) + \gamma\tilde{s} = 0, & x \in (0, L), \quad t > 0, \\ \tilde{w}(0, t) = \tilde{\xi}(0, t) = \tilde{s}(0, t) = 0, & t > 0, \\ \tilde{w}_{tt}(L, t) + \tilde{w}_x(L, t) + \tilde{\xi}(L, t) + \tilde{s}(L, t) = 0, & t > 0, \\ \tilde{\xi}_{tt}(L, t) + \tilde{\xi}_x(L, t) = 0, & t > 0, \\ \tilde{s}_{tt}(L, t) + \tilde{s}_x(L, t) = 0, & t > 0, \\ (\tilde{w}, \tilde{\xi}, \tilde{s})(x, 0) = (w_0, \xi_0, s_0)(x), & x \in (0, L), \\ (\tilde{w}_t, \tilde{\xi}_t, \tilde{s}_t)(x, 0) = (w_1, \xi_1, s_1)(x), & x \in (0, L), \end{cases}$$

given by Theorem 2.2. Thus, (z, η, φ) solves the following problem

$$\begin{cases} \rho_1 z_{tt} - k(z_x + \eta + \varphi)_x = F, & x \in (0, L), \quad t > 0, \\ \rho_2 \eta_{tt} - b\eta_{xx} + k(z_x + \eta + \varphi) = G, & x \in (0, L), \quad t > 0, \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + 3k(z_x + \eta + \varphi) + \gamma\varphi = H, & x \in (0, L), \quad t > 0, \\ z(0, t) = \eta(0, t) = \varphi(0, t) = 0, & t > 0, \\ z_{tt}(L, t) + z_x(L, t) + \eta(L, t) + \varphi(L, t) = 0, & t > 0, \\ \eta_{tt}(L, t) + \eta_x(L, t) = 0, & t > 0, \\ \varphi_{tt}(L, t) + \varphi_x(L, t) = 0, & t > 0, \\ (z, \eta, \varphi)(x, 0) = (0, 0, 0), & x \in (0, L), \\ (z_t, \eta_t, \varphi_t)(x, 0) = (0, 0, 0), & x \in (0, L), \end{cases} \quad (2.13)$$

where

$$\begin{cases} F(t, x) = \rho_1 u_1^{(2)}(t) \phi_1(x) - k \left[u_1(t) \phi_1^{(2)}(x) + (u_2(t) + u_3(t)) \phi_1'(x) \right], \\ G(t, x) = \rho_2 u_2^{(2)}(t) \phi_2(x) - b u_2(t) \phi_2^{(2)}(x) + k \left[u_1(t) \phi_1'(x) + (u_2(t) + u_3(t)) \phi_1(x) \right], \\ H(t, x) = \rho_2 u_3^{(2)}(t) \phi_3(x) - b u_3(t) \phi_3^{(2)}(x) + 3k \left[u_1(t) \phi_1'(x) + (u_2(t) + u_3(t)) \phi_1(x) \right] \\ + \gamma u_3(t) \phi_1(x), \end{cases}$$

and

$$\begin{pmatrix} F(t, x) \\ G(t, x) \\ H(t, x) \end{pmatrix} \in [C^1([0, \infty] \times [0, L])]^3.$$

Observe that, by employing the same strategy as in the previous section, system (2.13) can be reformulated as the following abstract evolution equation:

$$\begin{cases} W_t + \mathcal{A}W = \mathcal{K}, \\ W(0) = 0, \end{cases}$$

where

$$W = (z, z_t, \eta, \eta_t, \varphi, \varphi_t, z_t(L, T), \eta_t(L, t), \varphi_t(L, t))^T,$$

and

$$\mathcal{K} = \left(0, \frac{F}{\rho_1}, 0, \frac{G}{\rho_2}, 0, \frac{H}{\rho_2}, 0, 0, 0 \right)^T \in [C^1([0, \infty] \times [0, L])]^9.$$

Finally, since $\mathcal{A}$ generates a group of isometries in $\mathcal{H}$, it follows that system (2.13) admits a unique solution

$$W \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H}).$$

Reverting the change of variables in (2.12), we thus complete the proof. $\square$

Using the previous well-posedness results for the full system, we will study solutions of the system (1.3) in the transposition sense.

Definition 2.1. Given $T > 0$, $U_0 \in \mathcal{H}'$, $(h_1, h_2, h_3) \in \left(L^2 \left(0, T; (H_*^1(0, L))^* \right) \right)^3$ and $u_i \in L^2(0, T)$, for $i = 1, 2, 3$, let us consider the non-homogeneous system given by

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = h_1, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = h_2, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s = h_3, & x \in (0, L), \quad t \in (0, T), \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t \in (0, T), \\ w_{tt}(L, t) + w_x(L, t) + \xi(L, t) + s(L, t) = u_1(t), & t \in (0, T), \\ \xi_{tt}(L, t) + \xi_x(L, t) = u_2(t), & t \in (0, T), \\ s_{tt}(L, t) + s_x(L, t) = u_3(t), & t \in (0, T), \\ (w, \xi, s)(x, 0) = (w_0, \xi_0, s_0)(x), & x \in (0, L), \\ (w_t, \xi_t, s_t)(x, 0) = (w_1, \xi_1, s_1)(x), & x \in (0, L). \end{cases} \quad (2.14)$$

A vector function

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^T \in C([0, T]; D(\mathcal{A})),$$

is said to be a solution by transposition to (2.14) if the following identity holds

$$\begin{aligned} & \left\langle \begin{pmatrix} 3\rho_1 w_t(\cdot, \tau) \\ -3\rho_1 w(\cdot, \tau) \\ 3\rho_2 \xi_t(\cdot, \tau) \\ -3\rho_2 \xi(\cdot, \tau) \\ \rho_2 s_t(\cdot, \tau) \\ -\rho_2 s(\cdot, \tau) \\ -3kw(L, \tau) \\ -3b\xi(L, \tau) \\ -bs(L, \tau) \end{pmatrix}, \begin{pmatrix} \chi_0^\tau \\ \chi_1^\tau \\ \eta_0^\tau \\ \eta_1^\tau \\ \Theta_0^\tau \\ \Theta_1^\tau \\ \chi_t(L, \tau) \\ \eta_t(L, \tau) \\ \Theta_t(L, \tau) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 3\rho_1 w_1 \\ -3\rho_1 w_0 \\ 3\rho_2 \xi_1 \\ -3\rho_2 \xi_0 \\ \rho_2 s_1 \\ -\rho_2 s_0 \\ -3kw(L, 0) \\ -3b\xi(L, 0) \\ -bs(L, 0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi_t(L, 0) \\ \eta_t(L, 0) \\ \Theta_t(L, 0) \end{pmatrix} \right\rangle \\ & + \int_0^\tau \langle (3h_1(t), 3h_2(t), h_3(t)), (\chi(t), \eta(t), \Theta(t)) \rangle \left[(H_*^1(0, L))^* \right]_{H_*^1(0, L)}^3 dx dt \\ & + ((3ku_1(t), 3bu_2(t), bu_3(t)), 1_{(0, \tau)}(\chi(L, t), \eta(L, t), \Theta(L, t)))_{[L^2(0, T)]^3} \\ & + ((3kw_t(L, 0), 3b\xi_t(L, 0), bs_t(L, 0), (\chi(L, 0), \eta(L, 0), \Theta(L, 0))))_{\mathbb{R}^3} \end{aligned} \quad (2.15)$$

for any $\tau \in [0, T]$ and $W^\tau \in \mathcal{H}$, with

$$W^\tau = (\chi_0^\tau, \chi_1^\tau, \eta_0^\tau, \eta_1^\tau, \Theta_0^\tau, \Theta_1^\tau, \chi_1^\tau(L), \eta_1^\tau(L), \Theta_1^\tau(L))^T,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket in $\mathcal{H}' \times \mathcal{H}$ and $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^T$ is solution of the following adjoint system

$$\begin{cases} \rho_1 \chi_{tt} - k(\chi_x + \eta + \Theta)_x = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \rho_2 \eta_{tt} - b\eta_{xx} + k(\chi_x + \eta + \Theta) = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \rho_2 \Theta_{tt} - b\Theta_{xx} + 3k(\chi_x + \eta + \Theta) + \gamma\Theta = 0, & x \in (0, L), \quad t \in (0, \tau), \\ \chi(0, t) = \eta(0, t) = \Theta(0, t) = 0, & t \in (0, \tau), \\ \chi_{tt}(L, t) + \chi_x(L, t) + \eta(L, t) + \Theta(L, t) = 0, & t \in (0, \tau), \\ \eta_{tt}(L, t) + \eta_x(L, t) = 0, & t \in (0, \tau), \\ \Theta_{tt}(L, t) + \Theta_x(L, t) = 0, & t \in (0, \tau), \\ (\chi, \eta, \Theta)(x, \tau) = (\chi_0^\tau, \eta_0^\tau, \Theta_0^\tau), & x \in (0, L), \\ (\chi_t, \eta_t, \Theta_t)(x, \tau) = (\chi_1^\tau, \eta_1^\tau, \Theta_1^\tau), & x \in (0, L). \end{cases} \quad (2.16)$$

The next theorem establishes the existence and uniqueness of solutions for the system (1.3) in the transposition sense.

Theorem 2.4. Let $T > 0$, $U_0 \in D(\mathcal{A})$, $(h_1, h_2, h_3) \in \left(L^2 \left(0, T; (H_*^1(0, L))^* \right) \right)^3$ and $u_i \in L^2(0, T)$, for $i = 1, 2, 3$. Then, there exists a unique solution

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^T \in C([0, T]; D(\mathcal{A})),$$

of system (2.14) which verifies (2.15).

Proof. Let $T > 0$ and $\tau \in [0, T]$. By semigroup theory, precisely, from Theorem 2.2, the unique solution W of system (2.16) is given by

$$W(t) = S(\tau - t)W^\tau,$$

where $S(\cdot)$ is the semigroup generate by $\mathcal{A}$. Moreover, there exists $C_T > 0$, such that

$$\|W(t)\|_{\mathcal{H}} \leq C_T \|W^\tau\|_{\mathcal{H}}, \quad \forall t \in [0, \tau]. \quad (2.17)$$

Let us define a functional Δ given by the right-hand side of (2.15), that is

$$\begin{aligned} \Delta(W^\tau) = & \langle V_0^1, W(0) \rangle_{\mathcal{H}' \times \mathcal{H}} \\ & + \int_0^\tau \langle (3h_1(t), 3h_2(t, h_3(t))), (\chi(t), \eta(t), \Theta(t)) \rangle \left[\left(H_{*(0,L)}^1 \right)^*, H_{*(0,L)}^1 \right]^3 dx dt \\ & + \left((3ku_1(t), 3bu_2(t), bu_3(t)), 1_{(0,\tau)}(\chi(L, t), \eta(L, t), \Theta(L, t)) \right)_{[L^2(0,T)]^3} \\ & + ((3kw_t(L, 0), 3b\xi_t(L, 0), bs_t(L, 0)), (\chi(L, 0), \eta(L, 0), \Theta(L, 0)))_{\mathbb{R}^3}, \end{aligned}$$

where

$$V_0^1 = (3\rho_1 w_1, -3\rho_1 w_0, 3\rho_2 \xi_1, -3\rho_2 \xi_0, \rho_2 s_1, -\rho_2 s_0, -3kw(L, 0), -3b\xi(L, 0), -bs(L, 0))^\top.$$

Note that Δ is linear. Moreover, from (2.17) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned} |\Delta(W^\tau)| \leq & \|V_0^1\|_{\mathcal{H}} \|W(0)\|_{\mathcal{H}} + C_T \|W(t)\|_{\mathcal{H}} \|(u_1, u_2, u_3)\|_{[L^2(0,T)]^3} \\ & + C_T \|W(t)\|_{C([0,\tau];\mathcal{H})} \|(h_1, h_2, h_3)\|_{\left(L^2\left(0,T;\left(H_{*(0,L)}^1\right)^*\right) \right)^3} \\ & + C_T \|W(t)\|_{C([0,\tau];\mathcal{H})} \|(w_t(L, 0), \xi_t(L, 0), s_t(L, 0))\|_{\mathbb{R}^3} \\ \leq & C_T \left(\|V_0^1\|_{\mathcal{H}} + \|(u_1, u_2, u_3)\|_{[L^2(0,T)]^3} + \|(h_1, h_2, h_3)\|_{\left(L^2\left(0,T;\left(H_{*(0,L)}^1\right)^*\right) \right)^3} \right. \\ & \left. + \|(w_t(L, 0), \xi_t(L, 0), s_t(L, 0))\|_{\mathbb{R}^3} \right) \|W^\tau\|_{\mathcal{H}}. \end{aligned}$$

Hence, we obtain that $\Delta \in \mathcal{L}(D(\mathcal{A}); \mathbb{R})$. Thus, from the Riesz representation theorem, we obtain the existence and uniqueness of $U \in D(\mathcal{A})$ satisfying (2.15). Finally, the fact that $U \in C([0, T]; D(\mathcal{A}))$ is analogous to performed in [32, Lemma 3.2.], so we omit the details. $\square$

3 | Controllability Properties

In this section, we investigate the boundary controllability properties of system (1.3). As mentioned in the introduction, the controllability problem is closely related to establishing an observability inequality. Accordingly, we divide this section into two parts. First, we provide a characterization of the controllability of the system under consideration and present a corresponding observability inequality for the adjoint system. Finally, using this inequality, we prove the main result of the article.

3.1 | Observability Inequality

We begin with the following characterization of a control that drives system (1.3) to zero. Such results are well-established for dispersive systems³ and in a general context, for details see [29]. In our context, the following lemma gives an equivalent condition for the exact controllability property.

Lemma 3.1. *The initial data $U_0 \in \mathcal{H}'$ is controllable to zero in time $T > 0$ with controls $u_i \in L^2(0, T)$, for $i = 1, 2, 3$, if and only if*

$$\begin{aligned}
 & - \left\langle \begin{pmatrix} 3\rho_1 w_1 \\ -3\rho_1 w_0 \\ 3\rho_2 \xi_1 \\ -3\rho_2 \xi_0 \\ \rho_2 s_1 \\ -\rho_2 s_0 \\ -3kw(L, 0) \\ -3b\xi(L, 0) \\ -bs(L, 0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi(L, 0) \\ \eta(L, 0) \\ \Theta(L, 0) \end{pmatrix} \right\rangle_{\mathbb{R}^3} \\
 & - ((3kw_t(L, 0), 3b\xi_t(L, 0)bs_t(L, 0), (\chi(L, 0), \eta(L, 0), \Theta(L, 0))))_{\mathbb{R}^3} \\
 & = 3k \int_0^T u_1(t)\chi(L, t)dt + 3b \int_0^T u_2(t)\eta(L, t)dt + b \int_0^T u_3(t)\Theta(L, t)dt.
 \end{aligned} \tag{3.1}$$

Here, $\langle \cdot, \cdot \rangle$ is the duality of $\mathcal{H}' \times \mathcal{H}$ and (χ, η, Θ) is any solution of the following adjoint system

$$\begin{cases}
 \rho_1 \chi_{tt} - k(\chi_x + \eta + \Theta)_x = 0, & x \in (0, L), \quad t > 0, \\
 \rho_2 \eta_{tt} - b\eta_{xx} + k(\chi_x + \eta + \Theta) = 0, & x \in (0, L), \quad t > 0, \\
 \rho_2 \Theta_{tt} - b\Theta_{xx} + 3k(\chi_x + \eta + \Theta) + \gamma\Theta = 0, & x \in (0, L), \quad t > 0, \\
 \chi(0, t) = \eta(0, t) = \Theta(0, t) = 0, & t > 0, \\
 \chi_{tt}(L, t) + \chi_x(L, t) + \eta(L, t) + \Theta(L, t) = 0, & t > 0, \\
 \eta_{tt}(L, t) + \eta_x(L, t) = 0, & t > 0, \\
 \Theta_{tt}(L, t) + \Theta_x(L, t) = 0, & t > 0, \\
 (\chi, \eta, \Theta)(x, T) = (\chi_0^T, \eta_0^T, \Theta_0^T)(x), & x \in (0, L), \\
 (\chi_t, \eta_t, \Theta_t)(x, T) = (\chi_1^T, \eta_1^T, \Theta_1^T)(x), & x \in (0, L),
 \end{cases} \tag{3.2}$$

with $(\chi_0^T, \chi_1^T, \eta_0^T, \eta_1^T, \Theta_0^T, \Theta_1^T, \chi_1^T(L), \eta_1^T(L), \Theta_1^T(L))^T \in \mathcal{H}$.

Proof. The relation (3.1) is obtained first, by multiplying the Equations (1.3)₁, (1.3)₂, and (1.3)₃ by 3χ , 3η and Φ , respectively, where (χ, η, ϕ) is solution of (3.2). So, integrating by parts and using the boundary conditions of (1.3), the relation (3.1) follows. $\square$

Since we are using the control duality theory of Dolecki and Russell [27] in the set-up of Lions [28], relation (3.1) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$, given by

$$\mathcal{J}(W_0) = \frac{1}{2} \int_0^T (|\chi(L, t)|^2 + |\eta(L, t)|^2 + |\Theta(L, t)|^2) dt + \langle W_0, U_0 \rangle_{\mathcal{H} \times \mathcal{H}'},$$

where $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^T$ is solution of the adjoint system (3.2).

With this in hand, our task is to prove the existence of a minimizer for $\mathcal{J}$. It is well known that the existence of a minimizer to the functional $\mathcal{J}$ follows from the following observability inequality

$$\|W(0)\|_{\mathcal{H}}^2 \leq C \int_0^T \left(|\chi(L, t)|^2 dt + |\eta(L, t)|^2 + |\Theta(L, t)|^2 \right) dt, \quad (3.3)$$

for any

$$(\chi_0^\tau, \chi_1^\tau, \eta_0^\tau, \eta_1^\tau, \Theta_0^\tau, \Theta_1^\tau, \chi_1^\tau(L), \eta_1^\tau(L), \Theta_1^\tau(L))^\top \in D(\mathcal{A}^*).$$

Here, $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^\top$ is solution of the adjoint system (3.2). Now, our efforts are to prove that this inequality holds. Before proving the inequality (3.3), let us prove intermediary inequalities related to the traces of the solutions of (2.2). The first one can be read as follows.

Proposition 3.1. *Let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L, t), \xi_t(L, t), s_t(L, t))^\top$ the solution of the system (2.2). Then, for any $T > 0$ and $s \in (\frac{1}{2}, 1)$, there exist a positive constant C such that the following estimate hold*

$$\begin{aligned} \|U(t)\|_{\mathcal{H}}^2 &\leq C \int_0^T \left(|w_t(L, t)|^2 dt + |\xi_t(L, t)|^2 + |s_t(L, t)|^2 \right) dt \\ &\quad + C \|(w, \xi, s)\|_{L^\infty(0, T; [H^s(0, L)]^3)}. \end{aligned} \quad (3.4)$$

Proof. In order to obtain estimate (3.4), we multiply the first equation in (2.1) by $3xw_x$ and integrate by parts on $(0, L) \times (0, T)$ to obtain

$$\begin{aligned} 0 &= \int_0^L \int_0^T (\rho_1 w_{tt} - k(w_x + \xi + s)_x) 3xw_x dt dx \\ &= 3\rho_1 \int_0^L [xw_t w_x]_0^T dx - 3k \int_0^T L(w_x(L, t) + \xi(L, t) + s(L, t))w_x(L, t) dt \\ &\quad + 3k \int_0^L \int_0^T (w_x + \xi + s)xw_{xx} dt dx + 3k \int_0^L \int_0^T (w_x + \xi + s)^2 dt dx \\ &\quad - 3k \int_0^L \int_0^T (w_x + \xi + s)(\xi + s) dt dx - 3\rho_1 \int_0^L \int_0^T xw_t w_{xt} dt dx. \end{aligned} \quad (3.5)$$

Since

$$\int_0^L \int_0^T xw_t w_{xt} dt dx = \frac{L}{2} \int_0^T w_t^2(L, t) dt - \frac{1}{2} \int_0^L \int_0^T w_t^2 dt dx,$$

by substituting the above identity in (3.5), we get

$$\begin{aligned} &3\rho_1 \int_0^L [xw_t w_x]_0^T dx - \frac{3\rho_1 L}{2} \int_0^T w_t^2(L, t) dt + \frac{3\rho_1}{2} \int_0^L \int_0^T w_t^2 dt dx \\ &- 3k \int_0^T L(w_x(L, t) + \xi(L, t) + s(L, t))w_x(L, t) dt + 3k \int_0^L \int_0^T (w_x + \xi + s)xw_{xx} dt dx \\ &+ 3k \int_0^L \int_0^T (w_x + \xi + s)^2 dt dx - 3k \int_0^L \int_0^T (w_x + \xi + s)(\xi + s) dt dx = 0. \end{aligned} \quad (3.6)$$

Now, we multiply the second equation in (2.1) by $3x\xi_x$, the third one by xs_x , integrate by parts on $(0, L) \times (0, T)$ and we have that

$$\begin{aligned} 0 &= \int_0^L \int_0^T (\rho_2 \xi_{tt} - b \xi_{xx} + k(w_x + \xi + s)) 3x \xi_x dt dx \\ &= 3\rho_2 \int_0^L [x \xi_t \xi_x]_0^T dx - \frac{3\rho_2 L}{2} \int_0^T \xi_t^2(L, t) dt + \frac{3\rho_2}{2} \int_0^L \int_0^T \xi_t^2 dt dx \\ &\quad + 3k \int_0^L \int_0^T (w_x + \xi + s) x \xi_x dt dx - \frac{3bL}{2} \int_0^T \xi_x^2(L, t) dt + \frac{3b}{2} \int_0^L \int_0^T \xi_x^2 dt dx, \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} 0 &= \int_0^L \int_0^T (\rho_2 s_{tt} - b s_{xx} + 3k(w_x + \xi + s)) x s_x dt dx \\ &= \rho_2 \int_0^L [x s_t s_x]_0^T dx - \frac{\rho_2 L}{2} \int_0^T s_t^2(L, t) dt + \frac{\rho_2}{2} \int_0^L \int_0^T s_t^2 dt dx \\ &\quad + 3k \int_0^L \int_0^T (w_x + \xi + s) x s_x dt dx - \frac{bL}{2} \int_0^T s_x^2(L, t) dt + \frac{\gamma L}{2} \int_0^T s^2(L, t) dt \\ &\quad + \frac{b}{2} \int_0^L \int_0^T s_x^2 dt dx - \frac{\gamma L}{2} \int_0^L \int_0^T s^2 dt dx, \end{aligned} \quad (3.8)$$

respectively.

A straightforward computation shows that

$$\begin{aligned} &3k \int_0^L \int_0^T (w_x + \xi + s)(x w_{xx} + x \xi_x + x s_x) dt dx \\ &= \frac{3kL}{2} \int_0^T (w_x(L, t) + \xi(L, t) + s(L, t))^2 dt - \frac{3k}{2} \int_0^L \int_0^T (w_x + \xi + s)^2 dt dx. \end{aligned} \quad (3.9)$$

Thus, adding the identities (3.6)–(3.8) and by taking into account (3.9), it follows that

$$\begin{aligned} &\frac{1}{2} \int_0^T \|U(t)\|_{\mathcal{H}}^2 dt \\ &= \frac{1}{2} \int_0^T ((3\rho_1 L + 3k) w_t^2(L, t) + (3\rho_2 L + 3b) \xi_t^2(L, t) + (\rho_2 L + b) s_t^2(L, t)) dt \\ &\quad - \int_0^L [3\rho_1 x w_t w_x + 3\rho_2 x \xi_t \xi_x + \rho_2 x s_t s_x]_0^T dx + \frac{bL}{2} \int_0^T (3\xi_x^2(L, t) + s_x^2(L, t)) dt \\ &\quad + 3k \int_0^L \int_0^T (w_x + \xi + s)(\xi + s) dt dx + \frac{3kL}{2} \int_0^T (w_x(L, t) + \xi(L, t) + s(L, t)) w_x(L, t) dt \\ &\quad - \frac{3kL}{2} \int_0^T (w_x(L, t) + \xi(L, t) + s(L, t)) (\xi(L, t) + s(L, t)) dt \\ &\quad - \frac{\gamma L}{2} \int_0^T s^2(L, t) dt + \gamma L \int_0^L \int_0^T s^2 dt dx. \end{aligned} \quad (3.10)$$

We need to analyze each term of the RHS of (3.10). First, applying the Young inequality, we have that

$$3k \int_0^L (w_x + \xi + s)(\xi + s)dx \leq \varepsilon \left(3k \|(w_x + \xi + s)(\cdot, t)\|^2 + 3b \|\xi_x(\cdot, t)\|^2 + b \|s_x(\cdot, t)\|^2 \right) + C_{\varepsilon, k} \|(\xi, s)(\cdot, t)\|_{[H^s(0, L)]^2}^2, \quad (3.11)$$

for any $s \geq 0$ and $\varepsilon > 0$. On the other hand, using the Young inequality again and from the Sobolev embedding⁴, we obtain a positive constant $C_{\varepsilon_1, k, b, L}$, such that

$$\begin{aligned} & \frac{1}{2} \left| \left(bL(3\xi_x^2(L, t) + s_x^2(L, t)) + 3kL(w_x(L, t) + \xi(L, t) + s(L, t))w_x(L, t) \right. \right. \\ & \quad \left. \left. - 3kL(w_x(L, t) + \xi(L, t) + s(L, t))(\xi(L, t) + s(L, t)) \right) \right| \\ & \leq \varepsilon_1 \left(3k \|(w_x + \xi + s)(\cdot, t)\|^2 + 3b \|\xi_x(\cdot, t)\|^2 + b \|s_x(\cdot, t)\|^2 \right) \\ & \quad + C_{\varepsilon_1, k, b, L} \|(w, \xi, s)(\cdot, t)\|_{[H^s(0, L)]^3}^2, \end{aligned} \quad (3.12)$$

for any $\varepsilon_1 > 0$ and for $s \in (\frac{1}{2}, 1)$. Now, for all $t \in [0, T]$ and applying once more Young's inequality, we get that

$$\begin{aligned} \int_0^L |3\rho_1 x w_t w_x + 3\rho_2 x \xi_t \xi_x + \rho_2 x s_t s_x| dx & \leq \varepsilon_2 (3\rho_1 \|w_t(\cdot, t)\|^2 + 3\rho_2 \|\xi_t(\cdot, t)\|^2 \\ & \quad + \rho_2 \|s_t(\cdot, t)\|^2) + C_{\varepsilon_2, \rho_1, \rho_2, L} \|(w, \xi, s)(\cdot, t)\|_{[H^s(0, L)]^3}^2, \end{aligned} \quad (3.13)$$

for any $\varepsilon_2 > 0$, $s \in (\frac{1}{2}, 1)$ and some constant $C_{\varepsilon_2, \rho_1, \rho_2, L} > 0$.

With these estimates in hand, let us replace the inequalities (3.11)–(3.13) into (3.10) and pick $\varepsilon = \varepsilon_1 = \varepsilon_2$, it follows that

$$\begin{aligned} \frac{1}{2} \int_0^T \|U(t)\|_{\mathcal{H}}^2 dt & \leq 2\varepsilon \int_0^T \|U(t)\|_{\mathcal{H}}^2 dt + C \int_0^T (w_t^2(L, t) + \xi_t^2(L, t) + s_t^2(L, t)) dt \\ & \quad + C_\varepsilon \int_0^T \|(w, \xi, s)(\cdot, t)\|_{[H^s(0, L)]^3}^2 dt. \end{aligned}$$

From estimative above and we can choose ε small enough such that $0 < \varepsilon < \frac{1}{4}$ to deduce

$$\begin{aligned} \int_0^T \|U(t)\|_{\mathcal{H}}^2 dt & \leq C \int_0^T (w_t^2(L, t) + \xi_t^2(L, t) + s_t^2(L, t)) dt \\ & \quad + C \|(w, \xi, s)\|_{L^\infty(0, T; [H^s(0, L)]^3)}^2, \end{aligned} \quad (3.14)$$

for any $s \in (\frac{1}{2}, 1)$. Finally, using (2.11) and the inequality (3.14), we infer the estimative (3.4). This finishes the proof. $\square$

Let us now establish the second auxiliary inequality of this section.

Theorem 3.1. *Let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L, t), \xi_t(L, t), s_t(L, t))^T$ the solution of the system (2.2). Then, there exists a positive constant C such that the following estimate holds*

$$\|U_0\|_{\mathcal{H}}^2 \leq C \int_0^T (|w_t(L, t)|^2 + |\xi_t(L, t)|^2 + |s_t(L, t)|^2) dt. \quad (3.15)$$

Proof. Let us argue by contradiction, following the so-called “compactness-uniqueness” argument⁵. Suppose that (3.15) does not hold. Thus, there exists a sequence $\{U_0^n\}_{n \in \mathbb{N}} \in \mathcal{H}$, such that

$$1 = \|U_0^n\|_{\mathcal{H}}^2 > n \int_0^T \left(|w_t^n(L, t)|^2 dt + |\xi_t^n(L, t)|^2 + |s_t^n(L, t)|^2 \right) dt. \quad (3.16)$$

Note que (3.16) is equivalent to the next two assertions:

$$\|U^n(t)\|_{\mathcal{H}}^2 = \|U_0^n\|_{\mathcal{H}}^2 = 1, \quad \text{for all } t \in [0, T], \quad (3.17)$$

and

$$\int_0^T \left(|w_t^n(L, t)|^2 dt + |\xi_t^n(L, t)|^2 + |s_t^n(L, t)|^2 \right) dt \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.18)$$

where the function $U^n = (w^n, w_t^n, \xi^n, \xi_t^n, s^n, s_t^n, w_t^n(L, t), \xi_t^n(L, t), s_t^n(L, t))^T$ solves

$$\begin{cases} U_t^n(t) = \mathcal{A}U^n(t), \\ U^n(0) = U_0^n. \end{cases} \quad (3.19)$$

Observe that, from (3.17) and the definition of the norm, we get

$$\begin{cases} (w^n, \xi^n, s^n) & \text{bounded in } L^\infty(0, T; [H^1(0, L)]^3), \\ (w_t^n, \xi_t^n, s_t^n) & \text{bounded in } L^\infty(0, T; [L^2(0, L)]^3). \end{cases} \quad (3.20)$$

Since $H^1(0, L) \hookrightarrow H^s(0, L) \hookrightarrow L^2(0, L)$ and the embedding $H^1(0, L) \hookrightarrow H^s(0, L)$ is compact, for all $s \in (\frac{1}{2}, 1)$, from (3.20) and classical compactness results [33, Corollary 4], we can extract a subsequence of $\{(w^n, \xi^n, s^n)\}_{n \in \mathbb{N}}$, still denoted by $\{(w^n, \xi^n, s^n)\}_{n \in \mathbb{N}}$, such that

$$(w^n, \xi^n, s^n) \longrightarrow (w, \xi, s) \quad \text{strongly in } L^\infty(0, T; [H^s(0, L)]^3). \quad (3.21)$$

Thus, from (3.4), (3.18), and (3.21), we deduce that $\{U^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(0, T; \mathcal{H})$. Therefore,

$$U^n \longrightarrow U \quad \text{strongly in } L^\infty(0, T; \mathcal{H}),$$

and by equality (3.17), yields that

$$\|U(t)\|_{\mathcal{H}}^2 = 1, \quad \text{for all } t \in [0, T]. \quad (3.22)$$

Also,

$$\begin{aligned} 0 &= \liminf_{n \rightarrow 0} \left\{ \int_0^T \left(|w_t^n(L, t)|^2 dt + |\xi_t^n(L, t)|^2 + |s_t^n(L, t)|^2 \right) dt \right\} \\ &\geq \int_0^T \left(|w_t(L, t)|^2 dt + |\xi_t(L, t)|^2 + |s_t(L, t)|^2 \right) dt. \end{aligned} \quad (3.23)$$

From (3.23) and the above convergences, passing the limit in (3.19), we have that the limit $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L, t), \xi_t(L, t), s_t(L, t))^T$ satisfies the linear system

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = 0, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = 0, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s = 0, & x \in (0, L), \quad t \in (0, T), \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t \in (0, T), \\ w_x(L, t) + \xi(L, t) + s(L, t) = 0, & t \in (0, T), \\ \xi_x(L, t) = 0, & t \in (0, T), \\ s_x(L, t) = 0, & t \in (0, T). \end{cases} \quad (3.24)$$

Let $(\hat{w}, \hat{\xi}, \hat{s})$ the extension by zero of (w, ξ, s) , for $x \in (-a, a) \setminus (0, L)$, where $(0, L) \subset (-a, a)$ it is an interval. Then, $(\hat{w}, \hat{\xi}, \hat{s})$ solves, in $D'((0, L) \times (0, T))$, the following system

$$\begin{cases} \rho_1 \hat{w}_{tt} - k(\hat{w}_x + \hat{\xi} + \hat{s})_x = 0, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 \hat{\xi}_{tt} - b\hat{\xi}_{xx} + k(\hat{w}_x + \hat{\xi} + \hat{s}) = 0, & x \in (0, L), \quad t \in (0, T), \\ \rho_2 \hat{s}_{tt} - b\hat{s}_{xx} + 3k(\hat{w}_x + \hat{\xi} + \hat{s}) + \gamma \hat{s} = 0, & x \in (0, L), \quad t \in (0, T), \\ \hat{w}(0, t) = \hat{\xi}(0, t) = \hat{s}(0, t) = 0, & t \in (0, T), \\ \hat{w}_x(L, t) + \hat{\xi}(L, t) + \hat{s}(L, t) = 0, & t \in (0, T), \\ \hat{\xi}_x(L, t) = 0, & t \in (0, T), \\ \hat{s}_x(L, t) = 0, & t \in (0, T), \\ (\hat{w}, \hat{\xi}, \hat{s}) = (0, 0, 0), & x \in (-a, 0] \cup [L, a), \quad t \in (0, T). \end{cases} \quad (3.25)$$

Then, by Holmgren's uniqueness theorem,

$$(\hat{w}, \hat{\xi}, \hat{s}) \equiv (0, 0, 0) \text{ in } (-a, a) \times (0, T).$$

Hence, from (3.24) and (3.25), we have that $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L, t), \xi_t(L, t), s_t(L, t))^T$ satisfies

$$\begin{cases} k(w_x + \xi + s)_x = 0, & x \in (0, L), \quad t \in (0, T), \\ b\xi_{xx} - k(w_x + \xi + s) = 0, & x \in (0, L), \quad t \in (0, T), \\ bs_{xx} - 3k(w_x + \xi + s) - \gamma s = 0, & x \in (0, L), \quad t \in (0, T), \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t \in (0, T), \\ w_x(L, t) + \xi(L, t) + s(L, t) = 0, & t \in (0, T), \\ \xi_x(L, t) = 0, & t \in (0, T), \\ s_x(L, t) = 0, & t \in (0, T). \end{cases} \quad (3.26)$$

Thanks to the uniqueness of the solution to the previous system established by Lemma 2.1, the only solution to (3.26) is $U = 0$. However, this contradicts (3.22). Therefore, (3.15) must hold, which completes the proof of the proposition. $\square$

We are now in a position to prove the observability inequality.

Theorem 3.2. For any $U_0 \in \mathcal{H}$, let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L, t), \xi_t(L, t), s_t(L, t))^T$ the solution of the system (2.2). Then, there exists a positive constant C such that the following estimate holds

$$\|U_0\|_{\mathcal{H}}^2 \leq C \int_0^T \left(|w(L, t)|^2 + |\xi(L, t)|^2 + |s(L, t)|^2 \right) dt. \quad (3.27)$$

Proof. In order to prove the inequality (3.27), we consider the problem

$$\begin{cases} \tilde{U}_t = \mathcal{A}\tilde{U}, \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \quad (3.28)$$

where $\tilde{U}_0 = \mathcal{A}^{-1}U_0$. It is important to point out that the existence of $\mathcal{A}^{-1}$ is guaranteed by Lemma 2.1. Thus, from Theorem 3.1, the solution

$$\tilde{U} = (\tilde{w}, \tilde{w}_t, \tilde{\xi}, \tilde{\xi}_t, \tilde{s}, \tilde{s}_t, \tilde{w}_t(L, t), \tilde{\xi}_t(L, t), \tilde{s}_t(L, t))^{\top}$$

of the system (3.28) satisfies

$$\|\tilde{U}_0\|_{\mathcal{H}}^2 \leq C \int_0^T \left(|\tilde{w}_t(L, t)|^2 dt + |\tilde{\xi}_t(L, t)|^2 + |\tilde{s}_t(L, t)|^2 \right) dt, \quad (3.29)$$

for some constant $C > 0$.

On the other hand, it is straightforward to see that $\tilde{U}_t = U$. Then, we have that

$$(\tilde{w}_t, \tilde{\xi}_t, \tilde{s}_t) = (w, \xi, s). \quad (3.30)$$

Finally, from (3.29), (3.30) and by Riesz representation theorem, we deduce that

$$\begin{aligned} \|U_0\|_{\mathcal{H}}^2 &\leq C \|\tilde{U}_0\|_{\mathcal{H}}^2 \\ &\leq C \int_0^T \left(|\tilde{w}_t(L, t)|^2 dt + |\tilde{\xi}_t(L, t)|^2 + |\tilde{s}_t(L, t)|^2 \right) dt \\ &= C \int_0^T \left(|w(L, t)|^2 dt + |\xi(L, t)|^2 + |s(L, t)|^2 \right) dt, \end{aligned}$$

and the proof is achieved. $\square$

Due to the previous proposition, the following consequence holds.

Corollary 3.1. *Let $W = (\chi, \chi_t, \Theta, \Theta_t, \eta, \eta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^{\top}$ is solution of the adjoint system (3.2). Then, there exists a constant $C > 0$, such that the following observability inequality holds*

$$\|W(0)\|_{\mathcal{H}}^2 \leq C \int_0^T \left(|\chi(L, t)|^2 dt + |\eta(L, t)|^2 + |\Theta(L, t)|^2 \right) dt. \quad (3.31)$$

3.2 | Proof of Theorem 1.1

As a direct consequence of the observability inequality established in Corollary 3.1, combined with the HUM method from control theory, we obtain the controllability of the laminated system with Ventcel boundary conditions. In fact, let us consider the homogeneous system (2.2) with initial data $U_0 \in H'$, and let U be its corresponding solution. To evaluate this initial state in terms of boundary observables, we introduce the functional $\mathcal{J} : \mathcal{H} \rightarrow \mathbb{R}$, given by

$$\mathcal{J}(W_0) = \frac{1}{2} \int_0^T \left(|\chi(L, t)|^2 dt + |\eta(L, t)|^2 + |\Theta(L, t)|^2 \right) dt + \langle W_0, U_0 \rangle_{\mathcal{H} \times \mathcal{H}'},$$

where $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L, t), \eta_t(L, t), \Theta_t(L, t))^{\top}$ is solution of the adjoint system.

Note that $\mathcal{J}$ is continuous and convex. Moreover, the coercivity of functional $\mathcal{J}$ follows immediately from the observability inequality (3.1). Thus, thanks to [34, Corollary 3.23], we have that $\mathcal{J}$ has a minimizer. Since $\mathcal{J}$ is strictly convex, it follows that the minimizer is unique. From these facts, defining $u_1 = \chi(L, t)$, $u_2 = \eta(L, t)$ and $u_3 = \Theta(L, t)$ the null controllability holds true. $\square$

4 | Final Comments

In this work, we address the boundary controllability of a laminated beam governed by a system of three coupled equations, subject to dynamic Venttsel-type boundary conditions. To establish the required observability inequality, we employ the multiplier technique (cf. [29]) along with the Hilbert uniqueness method (cf. [28]). The main challenge stems from the need to control a system of three beam equations using three boundary inputs. As this is the first study tackling this specific problem, our results pave the way for further investigations in both the control and stabilization of such systems. Let us point out some of them.

- *Less controls:* A natural question arising from our analysis concerns the possibility of reducing the number of boundary controls. Addressing controllability with one or two boundary controls would require substantially different analytical techniques and is therefore beyond the scope of the present paper. Nevertheless, we conjecture that the use of Carleman estimates for the linear operator defined in (2.3) could allow the elimination of one boundary control. This question remains open.
- *Rapid stabilization:* It is well known in control theory that

$$\text{observability} \iff \text{controllability} \quad \text{and} \quad \text{observability} \implies \text{rapid stabilizability}$$

We believe that an application of an Ingham-type theorem (cf. [30]) to the system (1.3) could lead to a sharper observability inequality, which in turn would support the derivation of faster stabilization results. This remains another important issue to be explored.

- *Internal or distributed controls and Carleman estimates:* The techniques developed in this paper are specifically tailored to boundary controls acting through Venttsel-type boundary conditions. While fewer, or purely internal, controls can often be achieved by means of Carleman estimates, which provide a powerful and insightful framework, the corresponding observability inequalities would be substantially different and would likely require alternative tools. Employing Carleman estimates is therefore particularly pertinent, as such an approach could potentially lead to alternative proofs of observability inequalities, allow controllability with fewer control mechanisms, or enable extensions to semilinear models. However, due to the strong coupling of the system and the presence of dynamic Venttsel-type boundary conditions, the derivation of suitable Carleman estimates is highly nontrivial and, to the best of our knowledge, remains an interesting open problem.
- *Dissipative boundary terms and stabilization:* We end by discussing the stabilization problem. If dissipative terms are introduced into the Venttsel-type boundary conditions, one may indeed expect energy decay properties. Depending on the nature and strength of the dissipation, uniform or polynomial decay rates could potentially be obtained. We mention that recently, the authors treated a case of boundary stabilization [35]. In this work, there is an interesting review of the literature about this kind of problem and potential open issues.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

It does not apply since no new data were created or analyzed in this study.

Endnotes

- ¹The name of Alexander Ventcel is often spelled in various ways, such as Wentzell. We refer to the work [6], where this type of boundary condition was first introduced.
- ²See, for instance, [36].
- ³See, for example, [37, 38].
- ⁴See, for instance, [39, 40].
- ⁵For details of this argument see, for instance, [28, 41].

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