

# Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

## **Control of Kawahara equation using flat outputs**

Roberto de A. Capistrano-Filho and Jandeilson Santos da Silva

2026    vol. 8    no. 2



# Control of Kawahara equation using flat outputs

Roberto de A. Capistrano-Filho and Jandeilson Santos da Silva

We focus on the linear Kawahara equation in a bounded domain, employing two boundary controls. The controllability of this system has been previously demonstrated over the past decade using the Hilbert uniqueness method, which involves proving an observability inequality, in general, shown via Carleman estimates. Here, we extend this understanding by achieving exact controllability within a space of analytic functions, employing the flatness approach, which is a new approach for higher-order dispersive systems. We also provide a class of reachable states (taking 0 as initial data) which are holomorphic in some disk around 0.

## 1. Introduction

**1.1. Background and literature review.** Water wave models have been studied in diverse fields, including hydraulic engineering, fluid mechanics, physics, and mathematics. These models are generally difficult to derive and complex to analyze for qualitative information about wave dynamics, making their study interesting and challenging. Researchers have used appropriate assumptions on amplitude, wavelength, wave steepness, and other factors to investigate asymptotic models for water waves and gain insights into the full water wave system. For rigorous justification of various asymptotic models for surface and internal waves, see, for instance, [Alvarez-Samaniego and Lannes 2008; Bona et al. 2008; Lannes 2013].

In an appropriate nondimensional form, water waves can be modeled as a free boundary problem of the incompressible, irrotational Euler equation. This involves two nondimensional parameters:  $\delta := h/\lambda$  and  $\varepsilon := a/h$ , where the water depth, the wavelength, and the amplitude of the free surface are respectively denoted by  $h$ ,  $\lambda$  and  $a$ . Additionally, the parameter  $\mu$ , known as the Bond number, measures the relative importance of gravitational forces compared to surface tension forces. Long waves, also known as shallow water waves, are characterized by the condition  $\delta \ll 1$ . Various long-wave approximations depend on the relationship between  $\varepsilon$  and  $\delta$ .

The discussion above suggests that, instead of relying on models with poor asymptotic properties, one can rescale the mentioned parameters to find systems that reveal asymptotic models for surface and internal waves, such as the Kawahara model. Specifically, by setting  $\varepsilon = \delta^4 \ll 1$ ,  $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$ , considering the critical Bond number  $\mu = \frac{1}{3}$ , the Kawahara equation is derived. This equation, first

---

Capistrano-Filho was partially supported by CAPES/PRINT grant number 88887.311962/2018-00, CAPES/COFECUB grant number 88887.879175/2023-00, CNPq grant numbers 421573/2023-6, 307808/2021-1 and 401003/2022-1, and Propesqi (UFPE). Da Silva acknowledges support from CNPq.

MSC2020: 37L50, 93B05.

Keywords: Kawahara equation, controllability to the trajectories, flatness approach, Gevrey class, smoothing effect.

introduced by Hasimoto [1970] and Kawahara [1972], takes the form

$$\pm 2v_t + 3vv_x - v v_{xxx} + \frac{1}{45} v_{xxxxx} = 0,$$

or, after rescaling,

$$v_t + \alpha v_x + \beta v_{xxx} - v_{xxxxx} + vv_x = 0.$$

This equation is also known as the fifth-order Korteweg–de Vries (KdV) equation [Boyd 1991] or the singularly perturbed KdV equation [Pomeau et al. 1988]. It describes a dispersive partial differential equation that encompasses various wave phenomena, such as magnetoacoustic waves in a cold plasma [Kakutani 1960], the propagation of long waves in a shallow liquid beneath an ice sheet [Iguchi 2007], and gravity waves on the surface of a heavy liquid [Cui et al. 2006], among others.

Significant efforts in recent decades have aimed to understand this model within various research frameworks. For instance, numerous studies have focused on analytical and numerical methods for solving the equation. These methods include the tanh-function method [Berloff and Howard 1997], extended tanh-function method [Biswas 2009], sine-cosine method [Yusufoğlu et al. 2008], Jacobi elliptic functions method [Hunter and Scheurle 1988], direct algebraic method and numerical simulations [Polat et al. 2006], decomposition methods [Kaya and Al-Khaled 2007], as well as variational iteration and homotopy perturbation methods [Jin 2009]. Another important research direction is the study of the Kawahara equation from the perspective of control theory, specifically addressing the boundary controllability problem [Glass and Guerrero 2009], which is our motivation.

In this context, we are interested in the boundary controllability issue of the Kawahara equation in a bounded domain. Precisely, we investigate the linear control problem

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0, & (x, t) \in (-1, 0) \times (0, T), \\ u(0, t) = u_x(0, t) = u_{xx}(0, t) = 0, & t \in (0, T), \\ u(-1, t) = h_1(t), \quad u_x(-1, t) = h_2(t), & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in (-1, 0), \end{cases} \quad (1-1)$$

where  $h_1, h_2$  are the controls input and  $u$  is the state function. This work addresses two main issues:

**Problem A** (null controllability). *Given an initial data  $u_0$  in a suitable space, is it possible to find control functions  $h_1, h_2$  such that the state solution  $u(x, T)$  of the system (1-1) equals 0?*

**Problem B** (reachable functions). *Can we find a space  $\mathcal{R}$  with the property that, if the final data  $u_1$  is in  $\mathcal{R}$ , then one can get control functions  $h_1, h_2$  such that the solution  $u$  of the system (1-1) with  $u_0 = 0$  satisfies  $u(\cdot, T) = u_1$ ?*

It is important to highlight several results related to control problems associated with the system (1-1). Regarding the analysis of the Kawahara equation in a bounded interval, pioneering work was done by Vasconcellos and da Silva [2008]. They studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the influence of a localized damping mechanism. The second contribution in this area was made by Araruna, Capistrano-Filho and Doronin [Araruna et al. 2012], who considered a generalized Kawahara equation in a bounded domain.

Glass and Guerrero [2009] considered the problem

$$\left\{ \begin{array}{ll} y_t + \alpha y_{5x} = \sum_{k=0}^3 a_k(x, t) \partial_x^k y + h(x, t), & (x, t) \in (0, 1) \times (0, T), \\ y(0, t) = v_1(t), \quad y(1, t) = v_2(t), \quad y_x(0, t) = v_3(t), & t \in (0, T), \\ y_x(1, t) = v_4(t), \quad y_{xx}(0, t) = v_5(t), & t \in (0, T), \\ y(x, 0) = y_0, & x \in (0, 1), \end{array} \right. \quad (1-2)$$

where  $\alpha > 0$  and  $y_0, h, v_1, \dots, v_5$  are given functions. Using a Carleman estimate, they demonstrated that the system (1-2) is null controllable in the energy space  $L^2(0, 1)$  using only the controls on the right side of the boundary,  $v_2$  and  $v_4$ , meaning  $v_1 = v_3 = v_5 = 0$ . However, the authors noted that the controllability properties might fail if the set of controls is altered. For example, if  $v_1 = v_2 = v_3 = v_4 = 0$ , meaning only  $v_5$  is used as a control input, controllability does not occur because the adjoint system associated with (1-2) may have unobservable solutions.

The internal controllability problem for the Kawahara equation with homogeneous boundary conditions has been addressed by Chen [2019]. Using Carleman estimates associated with the linear operator of the Kawahara equation with internal observation, a null controllability result was demonstrated when the internal control is effective in a subdomain  $\omega \subset (0, L)$ . In [Capistrano-Filho and Gomes 2021], the authors consider the Kawahara equation with an internal control  $f(t, x)$  and homogeneous boundary conditions; the equation is shown to be exactly controllable in  $L^2$ -weighted Sobolev spaces. Additionally, it is shown to be controllable by regions in the  $L^2$ -Sobolev space.

**1.2. Flatness approach and main results.** The flatness approach [Fliess et al. 1995; Petit and Rouchon 2001; Fliess and Join 2013], also known as differential flatness, represents a powerful concept in control theory and nonlinear system analysis. It characterizes certain dynamical systems where all states and inputs can be described as algebraic functions of a finite set of independent variables, referred to as flat outputs, and their derivatives. This approach simplifies the control and trajectory planning of complex systems. It has found extensive application across various partial differential equations, including the heat equation [Martin et al. 2014; 2016b; Chen and Rosier 2022], 1-dimensional parabolic equations [Martin et al. 2016a], the 1-dimensional Schrödinger equation [Martin et al. 2018], the linear KdV equation [Martin et al. 2019], and more recently, the linear Zakharov–Kuznetsov equation [Chen and Rosier 2020]. Here are the key aspects of the flatness approach:

- (1) *Flat outputs:* In a flat system, outputs (flat outputs) describe the system's entire state and input trajectories using algebraic relationships involving a finite number of their derivatives.
- (2) *Simplified control design:* By utilizing flat outputs, complex nonlinear control problems can be transformed into simpler linear problems. This transformation simplifies the design of control laws and facilitates the generation of desired trajectories.
- (3) *Trajectory planning:* The flatness approach enables systematic trajectory planning. Desired trajectories for flat outputs can be planned first, and then corresponding state and input trajectories can be computed using the flatness property.

- (4) *Real-world applications*: The flatness approach has been successfully applied across various fields, including robotics, aerospace, process control, and automotive systems. For instance, in robotics, it aids in trajectory design for manipulators and mobile robots.

It is crucial to recognize the advantages of the flatness approach. Typically, designing controls for nonlinear systems is complex; however, this approach mitigates this complexity. Furthermore, flatness offers a systematic approach to generate trajectories and control inputs and, as previously noted, is applicable across a broad spectrum of dynamical systems.

Nevertheless, it is important to acknowledge the limitations and challenges associated with this method. Identifying flat outputs for a specific system can be demanding and necessitates a profound understanding of the system's dynamics. Additionally, not all systems exhibit flatness, which restricts the universal applicability of this approach. In summary, the flatness approach provides a robust framework for controlling and analyzing nonlinear dynamical systems through the concept of flat outputs. It simplifies the design of control laws and trajectory planning, making it a valuable tool across various engineering applications.

The primary contribution of this paper is to advance the study of the control problem for the fifth-order dispersive system (1-1). Unlike recent works that utilize boundary controls and use the Hilbert uniqueness method, introduced by Lions [1988], to show control properties, this article solves Problems A and B using two control inputs via the flatness approach. Let us now present the main results of this work.

Motivated by the smoothing effect exhibited by (1-1) with free evolution ( $h_1 = h_2 = 0$ ), we explore the reachable sets within smooth function spaces, specifically *Gevrey spaces*. First, let us introduce the definition of these spaces.

**Definition** (Gevrey spaces). Given  $s_1, s_2 \geq 0$ , a function  $u : [a, b] \times [t_1, t_2] \rightarrow \mathbb{R}$  is said to be Gevrey of order  $s_1$  on  $[a, b]$  and  $s_2$  on  $[t_1, t_2]$  if  $u \in C^\infty([a, b] \times [t_1, t_2])$  and there exist positive constants  $C$ ,  $R_1$  and  $R_2$  such that

$$|\partial_x^n \partial_t^m y(x, t)| \leq C \frac{n!^{s_1}}{R_1^n} \frac{m!^{s_2}}{R_2^m} \quad \text{for all } n, m \geq 0 \text{ and all } (x, t) \in [a, b] \times [t_1, t_2].$$

The vector space of all functions on  $[a, b] \times [t_1, t_2]$  which are Gevrey of order  $s_1$  in  $x$  and  $s_2$  in  $t$  is denoted by  $G^{s_1, s_2}([a, b] \times [t_1, t_2])$ .

In this context, we investigate the null controllability problem associated with (1-1) using the *flatness approach* to establish controllability properties. The goal is to identify a set of functions in a Gevrey space that are null controllable and to demonstrate that, at some intermediate time  $\tau \in (0, T)$ , the solutions to the free evolution problem fall into this set. Specifically, the first main result of this work can be stated as follows.

**Theorem 1.1** (null controllability). *Let  $s \in [\frac{5}{2}, 5)$  and  $T > 0$  be given. For any  $u_0 \in L^2(-1, 0)$ , there exist control inputs  $h_1, h_2 \in G^s([0, T])$  such that the solution of (1-1) belongs to the class  $u \in C([0, T], L^2(-1, 0)) \cap G^{\frac{1}{2}s, s}([-1, 0] \times [\varepsilon, T])$  for all  $\varepsilon \in (0, T)$ , and satisfies  $u(\cdot, T) = 0$ .*

This result confirms that two flat outputs can be used to achieve null controllability, thereby addressing [Problem A](#) described at the beginning of this work. Now, to present our second main result, we need to introduce some notations. Given  $z_0 \in \mathbb{C}$  and  $R > 0$ , we denote by  $D(z_0, R)$  the open disk given by

$$D(z_0, R) = \{z \in \mathbb{C}; |z - z_0| < R\},$$

and we let  $H(D(z_0, R))$  denote the set of holomorphic functions on  $D(z_0, R)$ . Henceforth, we consider the operators

$$Pu = \partial_x u + \partial_x^3 u - \partial_x^5 u,$$

where  $P^0 = \text{Id}$  and  $P^n = P \circ P^{n-1}$  when  $n \geq 1$ . In this way, the Kawahara equation can be expressed as

$$\partial_t u + Pu = 0. \quad (1-3)$$

Using induction and the fact that  $\partial_t$  and  $P$  commute we see that, if  $u = u(x, t)$  satisfies (1-3), then

$$\partial_t^n u + (-1)^{n-1} P^n u = 0. \quad (1-4)$$

For every  $R > 1$ , we define the set

$$\mathcal{R}_R := \{u \in C([-1, 0]); \text{ there exists } z \in H(D(0, R)) \text{ such that } u = z|_{[-1, 0]} \text{ and } (\partial_x^j P^n u)(0) = 0, j = 0, 1, 2\}, \quad (1-5)$$

from now on called *a set of reachable states*. The following result is the second main result in this paper, answering [Problem B](#).

**Theorem 1.2.** *Let  $T > 0$ ,  $R_0 := 2 \cdot 12^{5^{-1}} \cdot e^{(5e)^{-1}} > 1$  and  $R > 2R_0$  be given. For every  $u_1 \in \mathcal{R}_R$ , there exist control inputs  $h_1, h_2 \in G^5([0, T])$  for which the solution  $u$  of (1-1) with  $u_0 = 0$  satisfies  $u(\cdot, T) = u_1$  and  $u \in G^{1,5}([-1, 0] \times [0, T])$ .*

**1.3. Further comments and outline.** Observe that [Theorem 1.2](#) ensures that for the linear Kawahara equation, any reachable state can likely be extended as a holomorphic function on some open set in  $\mathbb{C}$ . The reachable states corresponding to controllability of the trajectories are in  $G^{\frac{1}{2}}([-1, 0])$ , allowing them to be extended as functions in  $H(\mathbb{C})$ . In contrast, the reachable functions in [Theorem 1.2](#) do not need to be holomorphic over the entire set  $\mathbb{C}$ ; they can have poles outside  $D(0, R)$ .

**Remark.** (i) Our result is entirely linear and applies only to the linear system (1-1). Therefore, a natural extension is to consider the nonlinear problem, which includes the term  $uu_x$ . However, it is required to modify the method used here. We believe that the strategy used in [\[Laurent and Rosier 2020\]](#) could be adapted to our work, though this remains an open problem.

(ii) The set defined by (1-5) is an example of reachable functions, though they are not completely understood. We believe there are other sets for which [Theorem 1.2](#) remains valid. For instance, in [\[Chen and Rosier 2020\]](#), the authors presented another example for an extension of the KdV equation in a 2-dimensional case. Thus, it remains an open question to verify other sets where [Theorem 1.2](#) holds.

(iii) As previously mentioned, several authors have applied the strategy of this article to different systems. In our case, additional difficulties arise when dealing with a fifth-order operator in space, specifically  $Pu = \partial_x u + \partial_x^3 u - \partial_x^5 u$ . Consequently, addressing the smoothing properties is challenging because we are working with five sets of different regularities, and additional terms appear. Furthermore, the set (1-5), and consequently, Theorem 1.2, can be obtained using the strategies outlined in [Martin et al. 2019]. However, the Gevrey space level needs to be adjusted, and the operator's order must be carefully adapted to our specific case.

(iv) Finally, observe that our results are verified by the Benney–Lin type equation

$$\partial_t u + \partial_x u + \partial_x^3 u + \mu_0 \partial_x^4 u - \partial_x^5 u = 0, \quad (x, t) \in (-1, 0) \times (0, T),$$

with the same boundary conditions as in (1-1) and with  $\mu_0 > 0$ . This equation describes the evolution of 1-dimensional small but finite amplitude long waves in various physical systems in fluid dynamics; see [Benney 1966; Lin 1974]. The coefficient  $\mu_0 > 0$  introduces nonconservative dissipative effects to the dispersive Kawahara equation (1-1) (where  $\mu_0 = 0$ ), and thus it is sometimes referred to as the strongly dissipative Kawahara equation [Zhou 2019], the fifth-order Korteweg–de Vries equation [Zhao et al. 2018], or the generalized Kawahara equation [Chen et al. 2022]. For more details about the Benney–Lin type equation, we encourage the reader to see [Coclite and di Ruvo 2022].

Let us conclude our introduction with an outline of our work. Section 2 addresses Problem A by presenting the control result, which is a consequence of the flatness property and the smoothing effect of the Kawahara equation. In Section 3, we provide an example of a set that can be reached from 0 by the system (1-1). This result, stemming from the flatness property extended to the limit case  $s = 5$ , partially answers Problem B.

## 2. Controllability result

We want to prove the null controllability property for the system (1-1). Our initial objective is to show the flatness property. This property ensures that we can parametrize the solution of the system (1-1) using the “flat outputs”  $\partial_x^3 u(0, t)$  and  $\partial_x^4 u(0, t)$ . To prove it, we will examine the ill-posed system

$$\begin{cases} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (x, t) \in (-1, 0) \times (0, T), \\ u(0, t) = \partial_x u(0, t) = \partial_x^2 u(0, t) = 0, & t \in (0, T), \\ \partial_x^3 u(0, t) = y(t), \quad \partial_x^4 u(0, t) = z(t), & t \in (0, T). \end{cases} \quad (2-1)$$

Precisely, we will show that the solution of (2-1) belongs to  $G^{\frac{1}{5}s, s}([-1, 0] \times [\varepsilon, T])$  for all  $\varepsilon \in (0, T)$  and can be written in the form

$$u(x, t) = \sum_{j=0}^{\infty} f_j(x) y^{(j)}(t) + \sum_{j=0}^{\infty} g_j(x) z^{(j)}(t), \quad (x, t) \in [-1, 0] \times [\varepsilon, T], \quad (2-2)$$

where  $y, z \in G^s([0, T])$  for some  $s \geq 0$ , with  $y^{(j)}(T) = z^{(j)}(T) = 0$  for every  $j \geq 0$ . Here,  $f_j$  and  $g_j$  are called the generating functions and are constructed following the ideas introduced in [Martin et al. 2016a].

The smoothing effect is responsible for ensuring that, from time  $\varepsilon$  onward, the solution of the free evolution problem associated to (1-1) is Gevrey of order  $\frac{1}{2}$  in  $x$  and  $\frac{5}{2}$  in  $t$ . Finally, a result of unique continuation is used to ensure that this solution coincides with the one associated with (2-1), described in (2-2).

**2.1. Flatness property.** We need to prove that there exists a one-to-one correspondence between solutions of (2-1) and a certain space of smooth functions, which we will call the flatness property. We will often use Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{for all } n,$$

and the inequality

$$(n+m)! \leq 2^{n+m} n! m!. \quad (2-3)$$

For  $n \in \mathbb{N}$ ,  $p \in [1, \infty]$  and  $f \in W^{n,p}(-1, 0)$ , we write

$$\|f\|_p = \|f\|_{L^p(-1,0)} \quad \text{and} \quad \|f\|_{n,p} = \sum_{i=0}^n \|\partial_x^i f\|_p.$$

We will find a solution  $u$  for (2-1) in the form

$$u(x, t) = \sum_{j \geq 0} f_j(x) y^{(j)}(t) + \sum_{j \geq 0} g_j(x) z^{(j)}(t). \quad (2-4)$$

Here,  $y, z, f_j, g_j$  satisfies the following conditions:

(1)  $y, z \in G^s([0, T])$  with  $s \in (1, 5)$ .

(2)  $f_j, g_j \in L^\infty([-1, 0])$  with polynomial growths in the form

$$|f_j(x)| \leq 2^j \frac{|x|^{5j+r}}{(5j+r)!} \quad \text{and} \quad |g_j(x)| \leq 2^j \frac{|x|^{5j+r}}{(5j+r)!} \quad \text{for all } j \geq 0 \text{ and all } x \in [-1, 0],$$

for some  $r \in \{0, 1, 2, 3, 4\}$ .

(3)  $\partial_x^3 u(0, t) = y(t)$  and  $\partial_x^4 u(0, t) = z(t)$ .

Considering  $u$  as in (2-4) and assuming that we can differentiate term by term, we get that

$$\begin{aligned} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u &= \sum_{j \geq 0} f_j(x) y^{(j+1)}(t) + \sum_{j \geq 0} (f_{jx} + f_{j3x} - f_{j5x})(x) y^{(j)}(t) \\ &\quad + \sum_{j \geq 0} g_j(x) z^{(j+1)}(t) + \sum_{j \geq 0} (g_{jx} + g_{j3x} - g_{j5x})(x) z^{(j)}(t). \end{aligned}$$

Considering a new index  $l = j + 1$  gives us

$$\sum_{j \geq 0} f_j(x) y^{(j+1)}(t) = \sum_{l \geq 1} f_{l-1}(x) y^{(l)}(t) \quad \text{and} \quad \sum_{j \geq 0} g_j(x) z^{(j+1)}(t) = \sum_{l \geq 1} g_{l-1}(x) z^{(l)}(t).$$

Thus,

$$\begin{aligned} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u \\ = \sum_{j \geq 1} f_{j-1}(x) y^{(j)}(t) + \sum_{j \geq 1} (f_{jx} + f_{j3x} - f_{j5x})(x) y^{(j)}(t) + (f_{0x} + f_{03x} - f_{05x})(x) y(t) \\ + (g_{0x} + g_{03x} - g_{05x})(x) z(t) + \sum_{j \geq 1} g_{j-1}(x) z^{(j)}(t) + \sum_{j \geq 1} (g_{jx} + g_{j3x} - g_{j5x})(x) z^{(j)}(t). \end{aligned}$$

Hence, if we have

$$\begin{cases} f_{0x} + f_{03x} - f_{05x} = g_{0x} + g_{03x} - g_{05x} = 0, \\ f_{jx} + f_{j3x} - f_{j5x} = -f_{j-1}, \quad g_{jx} + g_{j3x} - g_{j5x} = -g_{j-1} \quad \text{for all } j \geq 1, \end{cases}$$

then

$$u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0.$$

Furthermore, imposing the conditions

$$\begin{cases} f_0(0) = f_{0x}(0) = f_{02x}(0) = f_{04x}(0) = 0, \quad f_{03x}(0) = 1, \\ f_j(0) = f_{jx}(0) = f_{j2x}(0) = f_{j3x}(0) = f_{j4x}(0) = 0 \quad \text{for all } j \geq 1, \end{cases}$$

and

$$\begin{cases} g_0(0) = g_{0x}(0) = g_{02x}(0) = g_{03x}(0) = 0, \quad g_{04x}(0) = 1, \\ g_j(0) = g_{jx}(0) = g_{j2x}(0) = g_{j3x}(0) = g_{j4x}(0) = 0 \quad \text{for all } j \geq 1, \end{cases}$$

we obtain

$$u(0, t) = \partial_x u(0, t) = \partial_x^2 u(0, t) = 0, \quad \partial_x^3 u(0, t) = y(t), \quad \partial_x^4 u(0, t) = z(t) \quad \text{for } t \in (0, T).$$

This leads us to define inductively the functions  $\{f_j\}_{j \geq 0}$  and  $\{g_j\}_{j \geq 0}$  as follows:

(1)  $f_0$  is the solution of the IBVP

$$f_{0x} + f_{03x} - f_{05x} = 0, \quad f_0(0) = f_{0x}(0) = f_{02x}(0) = f_{04x}(0) = 0, \quad f_{03x}(0) = 1, \quad (2-5)$$

and, for  $j \geq 1$ ,  $f_j$  is the solution to the IBVP

$$f_{jx} + f_{j3x} - f_{j5x} = -f_{j-1}, \quad f_j(0) = f_{jx}(0) = f_{j2x}(0) = f_{j3x}(0) = f_{j4x}(0) = 0. \quad (2-6)$$

(2)  $g_0$  is the solution of the IBVP

$$g_{0x} + g_{03x} - g_{05x} = 0, \quad g_0(0) = g_{0x}(0) = g_{02x}(0) = g_{03x}(0) = 0, \quad g_{04x}(0) = 1, \quad (2-7)$$

and, for  $j \geq 1$ ,  $g_j$  is the solution to the IVP

$$g_{jx} + g_{j3x} - g_{j5x} = -g_{j-1}, \quad g_j(0) = g_{jx}(0) = g_{j2x}(0) = g_{j3x}(0) = g_{j4x}(0) = 0. \quad (2-8)$$

We will prove that, given  $s \in (1, 5)$ , there exists a one-to-one correspondence between solutions of (2-1) and pairs of functions  $(y, z) \in G^s([0, T]) \times G^s([0, T])$ ; namely,

$$u \mapsto (\partial_x^3 y(0, \cdot), \partial_x^4 z(0, \cdot)).$$

In the sense of this bijection, we shall say that the system (2-1) is flat. Observe that an expression in terms of the families  $\{f_j\}_{j \geq 0}$  and  $\{g_j\}_{j \geq 0}$  for a solution  $u$  of (2-1) as in (2-4) must be unique, that is, if

$$u(x, t) = \sum_{j \geq 0} f_j(x) y^{(j)}(t) + \sum_{j \geq 0} g_j(x) z^{(j)}(t) \quad \text{and} \quad u(x, t) = \sum_{j \geq 0} f_j(x) \tilde{y}^{(j)}(t) + \sum_{j \geq 0} g_j(x) \tilde{z}^{(j)}(t),$$

with  $y, \tilde{y}, z, \tilde{z} \in G^s([0, T])$ , then  $y = \tilde{y}$  and  $z = \tilde{z}$ .

**Remark.** (1) Considering a toy model, that is,

$$\begin{cases} u_t - \partial_x^5 u = 0, & (x, t) \in (-1, 0) \times (0, T), \\ u(0, t) = \partial_x u(0, t) = \partial_x^2 u(0, t) = 0, & t \in (0, T), \\ u(-1, t) = h_1(t), \quad \partial_x u(-1, t) = h_2(t), & t \in (0, T), \\ u(x, 0) = u_0(x), \end{cases}$$

the first equation in the systems (2-5)–(2-7) do not have the first and third derivative terms. Then, direct computation gives

$$f_j(x) = \frac{x^{5j+3}}{(5j+3)!} \quad \text{and} \quad g_j(x) = \frac{x^{5j+4}}{(5j+4)!} \quad \text{for all } j \geq 0, \quad x \in [-1, 0].$$

(2) Returning to the full system, with terms of the first and third derivatives, we have that the solutions  $f_0$  and  $g_0$  of (2-5) and (2-7) are given by

$$f_0(x) = \frac{1}{\sqrt{a}(a+b)} \sinh \sqrt{a}x - \frac{1}{\sqrt{b}(a+b)} \sin \sqrt{b}x \quad (2-9)$$

and

$$g_0(x) = \frac{1}{a(a+b)} \cosh \sqrt{a}x + \frac{1}{b(a+b)} \cos \sqrt{b}x - \frac{1}{a(a+b)} - \frac{1}{b(a+b)}, \quad (2-10)$$

respectively, where  $a = \frac{1}{2}(\sqrt{5} + 1)$  and  $b = \frac{1}{2}(\sqrt{5} - 1)$ .

To conclude that the system (2-1) is flat, it is enough to show that the solutions of (2-1) can be expressed as in (2-4) with  $\{f_j\}_{j \geq 0}$  and  $\{g_j\}_{j \geq 0}$  given by (2-5)–(2-8). Precisely, we will see that given  $y, z \in G^s([0, T])$ , then  $u$  given by (2-4) is well defined, belongs to  $G^{\frac{1}{5}s, s}([-1, 0] \times [0, T])$ , and it solves the problem

$$\begin{cases} u_t + \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & (x, t) \in (-1, 0) \times (0, T), \\ u(0, t) = \partial_x u(0, t) = \partial_x^2 u(0, t) = 0, & t \in (0, T), \\ \partial_x^3 u(0, t) = y(t), \quad \partial_x^4 u(0, t) = z(t) & t \in (0, T). \end{cases} \quad (2-11)$$

To do this, first, we need to establish estimates for the norms  $\|f_j\|_{L^\infty(-1, 0)}$  and  $\|g_j\|_{L^\infty(-1, 0)}$ , as suggested before. At this point, it is very useful to note that, for  $j \geq 1$ ,  $f_j$  (respectively  $g_j$ ) can be written in terms of  $f_0$  and  $f_{j-1}$  (respectively  $g_0$  and  $g_{j-1}$ ).

**Lemma 2.1.** *For any  $j \geq 1$  and  $x \in [-1, 0]$ , we have*

$$f_j(x) = \int_0^x \int_0^y f_0(y - \xi) f_{j-1}(\xi) d\xi dy \quad \text{and} \quad g_j(x) = \int_0^x g_0(x - \xi) g_{j-1}(\xi) d\xi. \quad (2-12)$$

*Proof.* Let  $x \in [-1, 0]$ ,  $y \in [x, 0]$  and  $\xi \in [y, 0]$  be given. From (2-6), we have

$$f_{j\xi}(\xi)f_0(y-\xi) + f_{j3\xi}(\xi)f_0(y-\xi) - f_{j5\xi}(\xi)f_0(y-\xi) = -f_0(y-\xi)f_{j-1}(\xi). \quad (2-13)$$

Integrating with respect to  $\xi$ , we obtain

$$\int_0^y f_j(\xi)(f_{0\xi}(y-\xi) + f_{03\xi}(y-\xi) - f_{05\xi}(y-\xi)) d\xi - f_{j\xi}(y) = -\int_0^y f_0(y-\xi)f_{j-1}(\xi) d\xi,$$

and by (2-5), after some integration by parts, it follows that

$$f'_j(y) = \int_0^y f_0(y-\xi)f_{j-1}(\xi) d\xi.$$

Integrating from 0 to  $x$  with respect to  $y$ , we get

$$f_j(x) = \int_0^x \int_0^y f_0(y-\xi)f_{j-1}(\xi) d\xi dy \quad \text{for all } x \in [-1, 0],$$

showing the left equality in (2-12). Similarly, the right equality in (2-12) is verified.  $\square$

The next lemma will ensure that the series in (2-4) are convergent.

**Lemma 2.2.** *For every  $j \geq 0$ , we have*

$$|f_j(x)| \leq 2^j \frac{|x|^{5j+1}}{(5j+1)!} \quad \text{and} \quad |g_j(x)| \leq 2^j \frac{|x|^{5j+1}}{(5j+1)!} \quad \text{for all } j \geq 0 \text{ and } x \in [-1, 0]. \quad (2-14)$$

*Proof.* Let us prove the left inequality in (2-14) by induction on  $j$ . First, we must verify the case  $j = 0$ :

$$|f_0(x)| \leq |x| \quad \text{for all } x \in [-1, 0].$$

Note that  $f_0(x) \leq 0$  in  $[-1, 0]$ . Indeed, from (2-9), since  $\cosh > 1$  in  $\mathbb{R} \setminus \{0\}$  and  $\cos \leq 1$  in  $\mathbb{R}$ ,

$$f_{0x} = \frac{1}{a+b} \cosh \sqrt{a}x - \frac{1}{a+b} \cos \sqrt{b}x > \frac{1}{a+b} - \frac{1}{a+b} = 0 \quad \text{for all } x \in (-\infty, 0). \quad (2-15)$$

This implies that  $f_0$  is increasing, thus  $f_0(x) \leq f_0(0) = 0$ . Therefore, it is sufficient to show that

$$f_0(x) \geq x \quad \text{for all } x \in [-1, 0]. \quad (2-16)$$

Defining  $\varphi(x) = f_0(x) - x$ , we have  $\varphi'(x) = f'_0(x) - 1$ . On the other hand, we have

$$f_{03x}(x) = \frac{a}{a+b} \cosh \sqrt{a}x + \frac{b}{a+b} \cos \sqrt{b}x > \frac{a-b}{a+b} = \frac{1}{\sqrt{5}} > 0 \quad \text{for } x \in (-\infty, 0),$$

and so  $f_{02x}$  is increasing in  $(-\infty, 0]$ , implying that  $f_{02x}(x) < f_{02x}(0) = 0$  for  $x \in (-\infty, 0]$ . Consequently  $f_{0x}$  is decreasing in  $(-\infty, 0]$ , which implies that

$$f_{0x}(x) < f_{0x}(-1) \cong 0.54 < 1 \quad \text{for all } x \in (-1, 0),$$

so  $\varphi' < 0$  in  $(-1, 0)$ . Hence  $\varphi$  is decreasing on  $[-1, 0]$  and therefore  $\varphi(x) \geq \varphi(0) = 0$ , for  $x \in [-1, 0]$ , which implies (2-16).

For the next step, we need to verify the estimate

$$|f_{03x}(x)| \leq 2 \quad (2-17)$$

when  $x \in [-1, 0]$ . Note that we have

$$f_{05x}(x) = \frac{a^2}{a+b} \cosh \sqrt{a}x - \frac{b^2}{a+b} \cos \sqrt{b}x > \frac{a^2 - b^2}{a+b} = a - b = 1 \quad \text{for } x \in (-\infty, 0).$$

Thus  $f_{04x}$  is increasing in  $(-\infty, 0]$ ; in particular,  $f_{04x}(x) < f_{04x}(0) = 0$ . Hence,  $f_{03x}$  is decreasing in  $(-\infty, 0]$ , and so  $1 = f_{03x}(0) \leq f_{03x}(x) \leq f_{03x}(-1) \cong 1.59 < 2$  for  $x \in [-1, 0]$ , yielding (2-17).

Now, suppose that for some  $j \geq 1$ ,

$$|f_j(x)| \leq 2^{j-1} \frac{|x|^{5(j-1)+1}}{[5(j-1)+1]!} \quad \text{for all } x \in [-1, 0].$$

By Lemma 2.1, we have

$$f_j(x) = \int_0^x \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi dy \quad \text{for all } x \in [-1, 0].$$

Using integration by parts (with respect to  $\xi$ ) together with the boundary conditions in (2-5), we obtain

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \int_0^y f_0(y-\xi) \frac{d}{d\xi} \int_0^\xi f_{j-1}(\sigma) d\sigma d\xi \\ &= \left[ f_0(y-\xi) \int_0^\xi f_{j-1}(\sigma) d\sigma \right]_0^y + \int_0^y \int_0^\xi f_{j-1}(\sigma) d\sigma f_{0\xi}(y-\xi) d\xi \\ &= \int_0^y f_{0\xi}(y-\xi) \int_0^\xi f_{j-1}(\sigma) d\sigma d\xi. \end{aligned}$$

Define  $F(\xi) = \int_0^\xi f_{j-1}(\sigma) d\sigma$ . Thus we can write

$$\int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi = \int_0^y f_{0\xi}(y-\xi) F(\xi) d\xi = \int_0^y f_{0\xi}(y-\xi) \frac{d}{d\xi} \int_0^\xi F(\tau) d\tau d\xi.$$

Integration by parts, together with the boundary conditions given in (2-5), yields

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \left[ f_{0\xi}(y-\xi) \int_0^\xi F(\tau) d\tau \right]_0^y + \int_0^y \int_0^\xi F(\tau) d\tau f_{02\xi}(y-\xi) d\xi \\ &= \int_0^y f_{02\xi}(y-\xi) \int_0^\xi F(\tau) d\tau d\xi. \end{aligned}$$

Setting also  $G(\xi) = \int_0^\xi F(\tau) d\tau$ , we have

$$\int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi = \int_0^y f_{02\xi}(y-\xi) G(\xi) d\xi = \int_0^y f_{02\xi}(y-\xi) \frac{d}{d\xi} \int_0^\xi G(\rho) d\rho d\xi.$$

Using again integration by parts together with (2-5), it follows that

$$\begin{aligned} \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \left[ f_{02\xi}(y-\xi) \int_0^\xi G(\rho) d\rho \right]_0^y + \int_0^y \int_0^\xi G(\rho) d\rho f_{03\xi}(y-\xi) d\xi \\ &= \int_0^y f_{03\xi}(y-\xi) \int_0^\xi G(\rho) d\rho d\xi. \end{aligned}$$

Therefore, the previous equality ensures that

$$\begin{aligned}
 \int_0^y f_0(y-\xi) f_{j-1}(\xi) d\xi &= \int_0^y f_{03\xi}(y-\xi) \int_0^\xi G(\rho) d\rho d\xi \\
 &= \int_0^y f_{03\xi}(y-\xi) \int_0^\xi \int_0^\rho F(\tau) d\tau d\rho d\xi \\
 &= \int_0^y f_{03\xi}(y-\xi) \int_0^\xi \int_0^\rho \int_0^\tau f_{j-1}(\sigma) d\sigma d\tau d\rho d\xi \\
 &= \int_0^y \int_0^\xi \int_0^\rho \int_0^\tau f_{03\xi}(y-\xi) f_{j-1}(\sigma) d\sigma d\tau d\rho d\xi,
 \end{aligned}$$

so

$$f_j(x) = \int_0^x \int_0^y \int_0^\xi \int_0^\rho \int_0^\tau f_{03\xi}(y-\xi) f_{j-1}(\sigma) d\sigma d\tau d\rho d\xi dy \quad \text{for all } x \in [-1, 0].$$

Then, by the induction hypothesis, we get that, for  $x \in [-1, 0]$ ,

$$\begin{aligned}
 |f_j(x)| &\leq \int_0^x \int_0^y \int_0^\xi \int_0^\rho \int_0^\tau 2 \cdot 2^{j-1} \frac{|\sigma|^{5(j-1)+1}}{[5(j-1)+1]!} d\sigma d\tau d\rho d\xi dy \\
 &\leq 2^j \left| \int_0^x \int_0^y \int_0^\xi \int_0^\rho \int_0^\tau \frac{|\sigma|^{5j-4}}{(5j-4)!} d\sigma d\tau d\rho d\xi dy \right| \\
 &= 2^j \left| \int_0^x \int_0^y \int_0^\xi \int_0^\rho \int_0^\tau \frac{(-\sigma)^{5j-4}}{(5j-4)!} d\sigma d\tau d\rho d\xi dy \right|.
 \end{aligned}$$

Pick  $r = -\sigma$ , thus  $dr = -d\sigma$ , that is,

$$\int_0^\tau \frac{(-\sigma)^{5j-4}}{(5j-4)!} d\sigma = - \int_0^{-\tau} \frac{r^{5j-4}}{(5j-4)!} dr = \left[ -\frac{r^{5j-3}}{(5j-3)(5j-4)!} \right]_0^{-\tau} = -\frac{(-\tau)^{5j-3}}{(5j-3)!}.$$

Thus,

$$|f_j(x)| \leq 2^j \left| \int_0^x \int_0^y \int_0^\xi \int_0^\rho \frac{(-\tau)^{5j-3}}{(5j-3)!} d\tau d\rho d\xi dy \right|.$$

Now set  $r = -\tau$  to get

$$\int_0^\rho \frac{(-\tau)^{5j-3}}{(5j-3)!} d\tau = - \int_0^{-\rho} \frac{r^{5j-3}}{(5j-3)!} dr = \left[ -\frac{r^{5j-2}}{(5j-2)(5j-3)!} \right]_0^{-\rho} = -\frac{(-\rho)^{5j-2}}{(5j-2)!},$$

and consequently,

$$|f_j(x)| \leq 2^j \left| \int_0^x \int_0^y \int_0^\xi \frac{(-\rho)^{5j-2}}{(5j-2)!} d\rho d\xi dy \right|.$$

Proceeding with three more integrations, we obtain

$$|f_j(x)| \leq 2^j \left| \int_0^x \int_0^y \frac{(-\xi)^{5j-1}}{(5j-1)!} d\xi dy \right| \leq 2^j \left| \int_0^x \frac{(-y)^{5j}}{(5j)!} dy \right| \leq 2^j \left| \frac{(-x)^{5j+1}}{(5j+1)!} \right| \quad \text{for all } x \in [-1, 0],$$

showing the left inequality in (2-14). The right inequality holds similarly.  $\square$

We are now in a position to solve the system (2-11).

**Proposition 2.3.** *Let  $s \in (1, 5)$ ,  $T > 0$  and  $y, z \in G^s([0, T])$  be given. The function  $u(x, t)$  defined in (2-4) belongs to  $G^{\frac{1}{5}s, s}([-1, 0] \times [0, T])$  and it solves (2-11). In particular, the corresponding controls  $h_1(t) := u(-1, t)$  and  $h_2(t) := \partial_x u(-1, t)$  are Gevrey of order  $s$  on  $[0, T]$ .*

*Proof.* Let us briefly explain our strategy to prove this result. Consider  $m, n \geq 0$  and  $(x, t) \in [-1, 0] \times [0, T]$  as given. Our task is first to differentiate the series in (2-4) as

$$\partial_x^n \partial_t^m u(x, t) = \sum_{j \geq 0} \partial_x^n \partial_t^m (f_j(x) y^{(j)}(t)) + \sum_{j \geq 0} \partial_x^n \partial_t^m (g_j(x) z^{(j)}(t)). \quad (2-18)$$

With this, since  $m, n$  are arbitrary, we will prove that the series in (2-18) is uniformly convergent in  $[-1, 0] \times [0, T]$ , which will ensure that  $u \in C^\infty([-1, 0] \times [0, T])$ . To conclude that  $u \in G^{\frac{1}{5}s, s}([-1, 0] \times [0, T])$  we must prove that

$$|\partial_x^n \partial_t^m u(x, t)| \leq C \frac{n!^{\frac{1}{5}s} m!^s}{R_1^n R_2^m}$$

for some constants  $C, R_1, R_2 > 0$ .

Let us choose  $s \in (1, 5)$  and  $y, z \in G^s([0, T])$ . Then there exists  $M, R > 0$  such that

$$|y^{(j)}(t)| \leq M \frac{j!^s}{R^j} \quad \text{and} \quad |z^{(j)}(t)| \leq M \frac{j!^s}{R^j} \quad \text{for all } j \geq 0 \text{ and all } t \in [0, T]. \quad (2-19)$$

For  $k \geq 0$ , since  $\partial_t$  and  $P$  commute, we have that

$$\partial_t^m P^k (f_j(x) y^{(j)}(t)) = P^k (f_j(x)) y^{(j+m)}(t).$$

From (2-5) and (2-6), it follows that  $Pf_0 = 0$  and  $Pf_j = -f_{j-1}$ , for  $j \geq 1$ . We will split, into two cases, our analysis of  $P^k(f_j)$ ; namely,  $j - k \geq 0$  and  $j - k < 0$ . First, suppose  $j - k \geq 0$ . In this case,

$$P^k(f_j) = (-1)^1 P^{k-1}(f_{j-1}) = (-1)^2 P^{k-2}(f_{j-2}) = \dots = (-1)^{k-1} P^{k-(k-1)}(f_{j-(k-1)}),$$

or equivalently,

$$P^k(f_j) = (-1)^{k-1} P(f_{j-k+1}) = (-1)^{k-1} (-f_{j-k}) = (-1)^k f_{j-k}.$$

Secondly, assuming  $j - k < 0$ , we have  $k - j > 0$ , so  $k - j \geq 1$ . Hence,

$$P^k(f_j) = (-1)^1 P^{k-1}(f_{j-1}) = (-1)^2 P^{k-2}(f_{j-2}) = \dots = (-1)^j P^{k-j-1}(0) = 0.$$

Putting the last two computations together, we get that

$$P^k(f_j) = \begin{cases} (-1)^k f_{j-k} & \text{if } j - k \geq 0, \\ 0 & \text{if } j - k < 0, \end{cases} \quad (2-20)$$

and consequently,

$$\partial_t^m P^k (f_j(x) y^{(j)}(t)) = \begin{cases} (-1)^k f_{j-k}(x) y^{(j+m)}(t) & \text{if } j - k \geq 0, \\ 0 & \text{if } j - k < 0. \end{cases} \quad (2-21)$$

Assume  $j \geq k$ , that is,  $j - k \geq 0$ . Then thanks to the relation (2-21) and Lemma 2.2, the estimate

$$|\partial_t^m P^k (f_j(x) y^{(j)}(t))| \leq 2^{j-k} \frac{|x|^{5(j-k)+1}}{[5(j-k)+1]!} M \frac{(j+m)!^s}{R^{j+m}} \leq M 2^{j-k} \frac{(j+m)!^s}{R^{j+m}} \frac{1}{(5(j-k)+1)!}$$

holds. Setting  $l = j - k$ , we can write the previous inequality as

$$|\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq M 2^l \frac{(l+k+m)!^s}{R^{l+k+m}} \frac{1}{(5l+1)!}.$$

Furthermore, writing  $N = k + m$  yields that

$$|\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq M 2^l \frac{(l+N)!^s}{R^{l+N}} \frac{1}{(5l+1)!}.$$

Then, using the inequality (2-3), we obtain that

$$|\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq M 2^l \frac{(2^l 2^N l! N!)^s}{R^l R^N} \frac{1}{(5l+1)!} = M \frac{2^{(1+s)l} 2^{Ns} l!^s N!^s}{R^l R^N} \frac{1}{(5l+1)!}. \quad (2-22)$$

Stirling's formula gives that

$$l! \sim \left(\frac{l}{e}\right)^l \sqrt{2\pi l} \quad \text{and} \quad (5l)! \sim \left(\frac{5l}{e}\right)^{5l} \sqrt{2\pi 5l}.$$

Observe that

$$\left(\frac{5l}{e}\right)^{5l} \sqrt{2\pi 5l} = 5^{5l+\frac{1}{2}} \left[\left(\frac{l}{e}\right)^l (\sqrt{2\pi l})\right]^5 (\sqrt{2\pi l})^{-4} \sim 5^{5l+\frac{1}{2}} (\sqrt{2\pi l})^{-4} l!^5$$

and, consequently,  $(5l)! \sim 5^{5l+\frac{1}{2}} (\sqrt{2\pi l})^{-4} l!^5$ . Thus, from (2-22), we get

$$|\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq \tilde{M} \frac{l!^s}{[2^{-(1+s)}R]^l} \frac{N!^s}{(2^{-s}R)^N} \frac{1}{(5l+1)5^{5l+\frac{1}{2}}(\sqrt{2\pi l})^{-4}l!^5},$$

for a suitable constant  $\tilde{M} > 0$ . Since  $1/((5l+1)5^{5l+\frac{1}{2}}) \leq 1$ , for  $l \geq 0$ , it follows that

$$|\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq \tilde{M} \frac{N!^s}{(2^{-s}R)^N} \frac{(2\pi l)^2}{[2^{-(1+s)}R]^l l!^{5-s}}.$$

Using inequality (2-3), we get that

$$\frac{N!^s}{(2^{-s}R)^N} \leq \frac{k!^s}{(R/4^s)^k} \frac{m!^s}{(R/4^s)^m},$$

and, consequently,

$$\sum_{j \geq k} \|\partial_t^m P^k(f_j y^{(j)}(t))\|_\infty \leq 4\tilde{M} \pi^2 \frac{k!^s}{(4^{-s}R)^k} \frac{m!^s}{(4^{-s}R)^m} \sum_{l \geq 0} \frac{l^2}{\tilde{R}^l l!^{5-s}},$$

for all  $m, k \geq 0$  and  $\tilde{R} = 2^{-(1+s)}R$ . To finish this case, observe that the series  $\sum_{l \geq 0} l^2/(\tilde{R}^l l!^{5-s})$  is convergent, so by the Weierstrass M-test, the series

$$\sum_{j \geq k} \|\partial_t^m P^k(f_j y^{(j)}(t))\|_\infty$$

is uniformly convergent on  $[-1, 0] \times [0, T]$ , for all  $m, k \geq 0$ . Furthermore,

$$\sum_{j \geq k} \|\partial_t^m P^k(f_j y^{(j)}(t))\|_\infty \leq \bar{M} \frac{k!^s}{(4^{-s}R)^k} \frac{m!^s}{(4^{-s}R)^m}, \quad (2-23)$$

for  $m, k \geq 0$ , where  $\bar{M} = 4\tilde{M} \pi^2 \sum_{l \geq 0} l^2/(\tilde{R}^l l!^{5-s})$ .

Analogously, it turns out that

$$P^k(g_j) = \begin{cases} (-1)^k g_{j-k} & \text{if } j-k \geq 0, \\ 0 & \text{if } j-k < 0, \end{cases} \quad (2-24)$$

and

$$\partial_t^m P^k(g_j(x)z^{(j)}(t)) = \begin{cases} (-1)^k g_{j-k}(x)z^{(j+m)}(t) & \text{if } j-k \geq 0, \\ 0 & \text{if } j-k < 0, \end{cases} \quad (2-25)$$

so using inequalities (2-3), (2-19), Stirling's formula and Lemma 2.1, we get

$$\sum_{j \geq k} \|\partial_t^m P^k(g_j z^{(j)}(t))\|_\infty \leq 4\tilde{M}\pi^2 \frac{k!^s}{(4^{-s}R)^k} \frac{m!^s}{(4^{-s}R)^m} \sum_{l \geq 0} \frac{l^2}{\tilde{R}^l l!^{5-s}} \quad \text{for all } m, k \geq 0.$$

Consequently, the series  $\sum \|\partial_t^m P^k(g_j z^{(j)}(t))\|_\infty$  is uniformly convergent in  $[-1, 0] \times [0, T]$  for all  $m, k \geq 0$ , with

$$\sum_{j \geq k} \|\partial_t^m P^k(g_j z^{(j)}(t))\|_\infty \leq \bar{M} \frac{k!^s}{(4^{-s}R)^k} \frac{m!^s}{(4^{-s}R)^m}. \quad (2-26)$$

Let  $K_3 > 0$  be as in Lemma 2.6 for  $p = \infty$ . From (2-21) and (2-23), we have

$$\sum_{j=0}^{\infty} \|\partial_t^m (f_j y^{(j)}(t))\|_{5i,\infty} = K_3^i \sum_{k=0}^i \sum_{j \geq k} \|\partial_t^m P^k(f_j y^{(j)}(t))\|_\infty \leq K_3^i \sum_{k=0}^i \bar{M} \frac{k!^s}{(4^{-s}R)^k} \frac{m!^s}{(4^{-s}R)^m}.$$

Thus,

$$\sum_{j=0}^{\infty} \|\partial_t^m (f_j y^{(j)}(t))\|_{5i,\infty} \leq K_3^i \bar{M} i!^s \left( \sum_{k=0}^i \frac{1}{(4^{-s}R)^k} \right) \frac{m!^s}{(4^{-s}R)^m}.$$

Defining  $b = 1 + 1/(4^{-s}R)$  and noting that

$$\sum_{j=0}^i \frac{1}{(4^{-s}R)^k} \leq \sum_{k=0}^i b^k = 1 \cdot \frac{b^{i+1} - 1}{b - 1} \leq \frac{b^{i+1}}{b - 1} = \frac{b}{b - 1} b^i,$$

yields

$$\sum_{j=0}^{\infty} \|\partial_t^m (f_j y^{(j)}(t))\|_{5i,\infty} \leq K_3^i \bar{M} i!^s \frac{b}{b - 1} b^i \frac{m!^s}{(4^{-s}R)^m}.$$

Hence, defining  $\hat{M} = \bar{M}b/(b - 1)$ , it follows that

$$\sum_{j=0}^{\infty} \|\partial_t^m (f_j y^{(j)}(t))\|_{5i,\infty} \leq \hat{M} \frac{i!^s}{(K_3^{-1}b^{-1})^i} \frac{m!^s}{(4^{-s}R)^m} \quad \text{for all } m, i \geq 0, t \in [0, T]. \quad (2-27)$$

Similarly, from (2-25), (2-26) and Lemma 2.6, we obtain

$$\sum_{j=0}^{\infty} \|\partial_t^m (g_j z^{(j)}(t))\|_{5i,\infty} \leq \hat{M} \frac{i!^s}{(K_3^{-1}b^{-1})^i} \frac{m!^s}{(4^{-s}R)^m} \quad \text{for all } m, i \geq 0, t \in [0, T]. \quad (2-28)$$

Given  $m, n \geq 0$ , consider  $i \geq 0$  such that  $n \in \{5i - r; r = 0, 1, 2, 3, 4\}$ . Then from (2-27) we get, for  $(x, t) \in [-1, 0] \times [0, T]$ , that

$$\sum_{j=0}^{\infty} |\partial_x^n \partial_t^m (f_j(x) y^{(j)}(t))| \leq \hat{M} \frac{i!^s}{(K_3^{-1} b^{-1})^i} \frac{m!^s}{(4^{-s} R)^m},$$

and from (2-28),

$$\sum_{j=0}^{\infty} |\partial_x^n \partial_t^m (g_j(x) z^{(j)}(t))| \leq \hat{M} \frac{i!^s}{(K_3^{-1} b^{-1})^i} \frac{m!^s}{(4^{-s} R)^m}.$$

By the Weierstrass M-test, it follows that these series are uniformly convergent in  $[-1, 0] \times [0, T]$  for all  $m, n \geq 0$ . Thus, defining  $u : [-1, 0] \times [0, T] \rightarrow \mathbb{R}$  as in (2-4), we have that  $\partial_x^n \partial_t^m u$  is continuous. Consequently  $u \in C^\infty([-1, 0] \times [0, T])$  satisfies

$$|\partial_x^n \partial_t^m u(x, t)| \leq 2\hat{M} \frac{i!^s}{(K_3^{-1} b^{-1})^i} \frac{m!^s}{(4^{-s} R)^m} \quad \text{for all } (x, t) \in [-1, 0] \times [0, T]. \quad (2-29)$$

As seen before, Stirling's formula gives us  $(5i)! \sim 5^{5i+\frac{1}{2}} (\sqrt{2\pi i})^{-4} i!^5 = 5^{5i+\frac{1}{2}} (2\pi i)^{-2} i!^5$ , and so  $i!^5 \sim (2\pi i)^2 (5i)! / (5^{5i+\frac{1}{2}})$ . Therefore  $i!^s \sim [(2\pi i)^2 (5i)! / (5^{5i+\frac{1}{2}})]^{\frac{1}{5}s}$ . Once we have that  $i^2 / (5^{5i+\frac{1}{2}}) \leq 1$  for all  $i \geq 0$ , we can infer that  $i!^s \leq M^* (4\pi^2)^{\frac{1}{5}s} (5i)!^{\frac{1}{5}s}$  for some constant  $M^* > 0$ . Furthermore,  $n = 5i - r$ , that is,  $5i = n + r$  with  $r \in \{0, 1, 2, 3, 4\}$ . Then, by (2-3),

$$i!^s \leq M^* (4\pi^2)^{\frac{1}{5}s} (2^n 2^r n! r!)^{\frac{1}{5}s} = M^* (4\pi^2 \cdot 2^r \cdot r!)^{\frac{1}{5}s} (2^n)^{\frac{1}{5}s} n!^{\frac{1}{5}s} \leq M' (2^{\frac{1}{5}s})^n n!^{\frac{1}{5}s},$$

where  $M' = M'(r) = M^* (4\pi^2 \cdot 2^r \cdot r!)^{\frac{1}{5}s}$ . With this, returning to inequality (2-29), we obtain

$$|\partial_x^n \partial_t^m u(x, t)| \leq 2\hat{M} M' (2^{\frac{1}{5}s})^n \frac{n!^{\frac{1}{5}s}}{(K_3^{-1} b^{-1})^{\frac{1}{5}(n+r)}} \frac{m!^s}{(4^{-s} R)^m} \leq \frac{2\hat{M} M'}{(K_3^{-1} b^{-1})^{\frac{1}{5}r}} \frac{n!^{\frac{1}{5}s}}{(K_3^{-\frac{1}{5}} b^{-\frac{1}{5}} 2^{-\frac{1}{5}s})^n} \frac{m!^s}{(4^{-s} R)^m}.$$

Finally, defining

$$M'' := \max \left\{ \frac{2\hat{M} M'}{(K_3^{-1} b^{-1})^{\frac{1}{5}r}}; r = 0, 1, 2, 3, 4 \right\}, \quad R_1 := K_3^{-\frac{1}{5}} b^{-\frac{1}{5}} 2^{-\frac{1}{5}s} \quad \text{and} \quad R_2 := 4^{-s} R,$$

we have

$$|\partial_x^n \partial_t^m u(x, t)| \leq M'' \frac{n!^{\frac{1}{5}s}}{R_1^n} \frac{m!^s}{R_2^m} \quad \text{for all } n, m \geq 0 \text{ and all } (x, t) \in [-1, 0] \times [0, T].$$

Moreover,  $u$  solves (2-11) by construction. The result follows.  $\square$

**2.2. Estimates in  $W^{5n,p}(-1, 0)$ -norm.** Let us start by remembering that the map

$$\|\cdot\|_* : W^{5,p}(-1, 0) \rightarrow \mathbb{R}_+, \quad f \mapsto \|f\|_* := \|f\|_p + \|Pf\|_p,$$

is a norm called the *graph* norm associated with the operator  $P : W^{5,p}(-1, 0) \rightarrow L^p(-1, 0)$ .

**Lemma 2.4.** *Let  $p \in [1, \infty]$  be given. For all  $n \geq 0$ , we have*

$$\|P^n f\|_p \leq 3^n \|f\|_{5n,p} \quad \text{for all } f \in W^{5n,p}(-1, 0). \quad (2-30)$$

*Proof.* For  $n = 0$ , the inequality is obvious. For  $n = 1$ , given  $f \in W^{5,p}(-1, 0)$  it follows that

$$\|Pf\|_p = \|\partial_x f + \partial_x^3 f - \partial_x^5 f\|_p \leq \|\partial_x f\|_p + \|\partial_x^3 f\|_p + \|\partial_x^5 f\|_p \leq 3\|f\|_{5,p}.$$

Suppose that (2-30) is satisfied for  $0, 1, \dots, n-1$ . Then for  $f \in W^{5n,p}(-1, 0)$ , we get

$$\|P^n f\|_p = \|P^{n-1} Pf\|_p \leq 3^{n-1} \|Pf\|_{5n-5,p} \leq 3^{n-1} \left( 3 \sum_{j=0}^{5n} \|\partial_x^j f\|_p \right) = 3^n \|f\|_{5n,p},$$

so (2-30) is true for  $n$ , which concludes the proof.  $\square$

**Lemma 2.5.** *Let  $p \in [1, \infty]$  be given. There exists a constant  $K_1 = K_1(p) > 0$  such that*

$$\|f\|_{5,p} \leq K_1(\|f\|_p + \|Pf\|_p) \quad \text{for all } f \in W^{5,p}(-1, 0).$$

*Proof.* We know that  $\|\cdot\|_*$  is a norm in  $W^{5,p}(-1, 0)$ . We will show that  $(W^{5,p}(-1, 0), \|\cdot\|_*)$  is a Banach space. To do this, consider a Cauchy sequence in  $(W^{5,p}(-1, 0), \|\cdot\|_*)$ ,  $(f_n)_{n \geq 0}$ . Then,

$$\|f_m - f_n\|_* = \|f_m - f_n\|_p + \|Pf_m - Pf_n\|_p \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Since  $L^p(-1, 0)$  is a Banach space, there exist  $f, g \in L^p(-1, 0)$  such that

$$\|f_n - f\|_p \rightarrow 0, \quad \|Pf_n - g\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Given  $\varphi \in C_0^\infty(-1, 0)$ , we have

$$\left| \int_{-1}^0 (f_n - f) \varphi \right| \leq \int_{-1}^0 |f_n - f| |\varphi| \leq \begin{cases} \|f_n - f\|_p \|\varphi\|_q, & 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f_n - f\|_1 \|\varphi\|_\infty, & p = 1, \\ \|f_n - f\|_\infty \|\varphi\|_1, & p = \infty. \end{cases}$$

Therefore,

$$\int_{-1}^0 (f_n - f) \varphi \rightarrow 0 \quad \text{for all } \varphi \in C_0^\infty(-1, 0),$$

which implies that

$$f_n \rightarrow f \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-31)$$

Analogously, we infer that  $Pf_n \rightarrow g$  in  $\mathcal{D}'(-1, 0)$ . But (2-31) implies that

$$\partial_x^i f_n \rightarrow \partial_x^i f \quad \text{in } \mathcal{D}'(-1, 0) \text{ for all } i \in \mathbb{N} \cup \{0\},$$

and consequently  $Pf_n \rightarrow Pf$  in  $\mathcal{D}'(-1, 0)$ . By the uniqueness of the limit, it follows that

$$Pf = g \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-32)$$

Consider  $T_1, T_2 \in \mathcal{D}'(-1, 0)$  given by  $T_1 = f + \partial_x^2 f - \partial_x^4 f$  and  $T_2 = h_1$ , where

$$h_1(x) = \int_{-1}^x g(t) dt.$$

Note that (2-32) implies that  $\partial_x T_1 = g$  in  $\mathcal{D}'(-1, 0)$ . On the other hand,  $h_1 \in L^p(-1, 0)$  and

$$\langle \partial_x T_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = - \int_{-1}^0 h_1(x) \varphi'(x) dx = - \int_{-1}^0 \left( \int_{-1}^x g(t) dt \right) \varphi'(x) dx = - \int_{-1}^0 \int_{-1}^x g(t) \varphi'(x) dt dx.$$

For  $-1 \leq t \leq x \leq 0$ , Fubini's theorem gives us

$$\langle \partial_x T_2, \varphi \rangle_{\mathcal{D}', \mathcal{D}} = - \int_{-1}^0 g(t)(\varphi(0) - \varphi(t)) dt = \int_{-1}^0 g(t)\varphi(t) dt = \langle g, \varphi \rangle_{\mathcal{D}', \mathcal{D}},$$

and therefore  $\partial_x T_2 = g = \partial_x T_1$  in  $\mathcal{D}'(-1, 0)$ . Consequently, there exists a constant  $k_1 \in \mathbb{R}$  such that  $T_1 = T_2 + k_1$  in  $\mathcal{D}'(-1, 0)$ , that is,

$$f + \partial_x^2 f - \partial_x^4 f = h_1 + k_1 \quad \text{in } \mathcal{D}'(-1, 0).$$

Defining  $\tilde{h}_1 = h_1 + k_1 - f$ , we have  $\tilde{h}_1 \in L^p(-1, 0)$  and

$$\partial_x^2 f - \partial_x^4 f = \tilde{h}_1 \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-33)$$

Now, consider  $T_3, T_4 \in \mathcal{D}'(-1, 0)$  given by  $T_3 = \partial_x f - \partial_x^3 f$  and  $T_4 = h_2$ , where  $h_2 \in L^p(-1, 0)$  is

$$h_2(x) = \int_{-1}^x \tilde{h}_1(t) dt.$$

Proceeding as before we obtain  $\partial_x T_3 = \tilde{h}_1 = \partial_x T_4$  in  $\mathcal{D}'(-1, 0)$ , thus there exists a constant  $k_2 \in \mathbb{R}$  such that  $T_3 = T_4 + k_2$  in  $\mathcal{D}'(-1, 0)$ , or equivalently,

$$\partial_x f - \partial_x^3 f = h_2 + k_2 \quad \text{in } \mathcal{D}'(-1, 0).$$

Defining  $\tilde{h}_2 := h_2 + k_2$ , we have  $\tilde{h}_2 \in L^p(-1, 0)$ , and

$$\partial_x f - \partial_x^3 f = \tilde{h}_2 \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-34)$$

Set  $T_5, T_6 \in \mathcal{D}'(-1, 0)$  as  $T_5 = f - \partial_x^2 f$  and  $T_6 = h_3$ , where  $h_3 \in L^p(-1, 0)$  is given by

$$h_3(x) = \int_{-1}^x \tilde{h}_2(t) dt.$$

By the same argument as before,

$$f - \partial_x^2 f = h_3 + k_3 \quad \text{in } \mathcal{D}'(-1, 0)$$

with  $k_3 \in \mathbb{R}$ . Defining  $\tilde{h}_3 = f - h_3 - k_3$ , we have  $\tilde{h}_3 \in L^p(-1, 0)$  and

$$\partial_x^2 f = \tilde{h}_3 \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-35)$$

Now define  $T_7, T_8 \in \mathcal{D}'(-1, 0)$  by  $T_7 = \partial_x f$  and  $T_8 = h_4$ , where  $h_4 \in L^p(-1, 0)$  is given by

$$h_4(x) = \int_{-1}^x \tilde{h}_3(t) dt.$$

Again, the same argument ensures that there exists  $k_4 \in \mathbb{R}$  such that, defining  $\tilde{h}_4 := h_4 + k_4$ , we have  $\tilde{h}_4 \in L^p(-1, 0)$  and

$$\partial_x f = \tilde{h}_4 \quad \text{in } \mathcal{D}'(-1, 0). \quad (2-36)$$

Since  $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4 \in L^p(-1, 0)$ , the equalities (2-32)–(2-36) give us that  $\partial_x f, \partial_x^2 f, \partial_x^3 f, \partial_x^4 f$  and  $\partial_x^5 f$  belong to  $L^p(-1, 0)$ , so that  $f \in W^{5,p}(-1, 0)$ . Furthermore,

$$\|f_n - f\|_* = \|f_n - f\|_p + \|Pf_n - Pf\|_p = \|f_n - f\|_p + \|Pf_n - g\|_p,$$

which implies that

$$\|f_n - f\|_* \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

that is,  $f_n \rightarrow f$  in  $(W^{5,p}(-1, 0), \|\cdot\|_*)$ , showing that  $(W^{5,p}(-1, 0), \|\cdot\|_*)$  is a Banach space.

Consider the map

$$I : (W^{5,p}(-1, 0), \|\cdot\|_{5,p}) \rightarrow (W^{5,p}(-1, 0), \|\cdot\|_*), \quad f \mapsto I(f) = f.$$

Note that  $I$  is linear and bijective and  $\|I(f)\|_* \leq \|f\|_{5,p}$ , so  $I$  is continuous. Thus,  $I^{-1}$  is also continuous. Therefore, there exists  $K_1 > 0$  such that

$$\|f\|_{5,p} \leq K_1(\|f\|_p + \|Pf\|_p) \quad \text{for all } f \in W^{5,p}(-1, 0),$$

showing the result.  $\square$

**Remark.** As a consequence of [Lemma 2.5](#), we see that the norms  $\|\cdot\|_*$  and  $\|\cdot\|_{5,p}$  are equivalent in the space  $W^{5,p}(-1, 0)$ .

**Lemma 2.6.** *Let  $p \in [1, \infty]$  be given. There exists a constant  $K_2 = K_2(p) > 0$  such that, for every  $n \geq 1$ ,*

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^n \|P^i f\|_p \leq \|f\|_{5n,p} \leq (1 + 2K_2)^{n-1} K_2 \sum_{i=0}^n \|P^i f\|_p \quad \text{for all } f \in W^{5n,p}(-1, 0).$$

Consequently,

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^n \|P^i f\|_p \leq \|f\|_{5n,p} \leq K_3^n \sum_{i=0}^n \|P^i f\|_p \quad \text{for all } f \in W^{5n,p}(-1, 0),$$

for every  $n \geq 0$ , where  $K_3 = K_3(p) = 1 + 2K_2$ .

*Proof.* Given  $n \geq 0$  and  $f \in W^{5n,p}(-1, 0)$ , we have  $f \in W^{5i,p}(-1, 0)$  and  $\|f\|_{5i,p} \leq \|f\|_{5n,p}$ ,  $0 \leq i \leq n$ . Then by [Lemma 2.4](#), we get

$$\sum_{i=0}^n \|P^i f\|_p \leq \sum_{i=0}^n 3^i \|f\|_{5i,p} \leq \left( \sum_{i=0}^n 3^i \right) \|f\|_{5n,p} = 1 \cdot \frac{3^{n+1}-1}{3-1} \|f\|_{5n,p} \leq \frac{3^{n+1}}{2} \|f\|_{5n,p},$$

and therefore,

$$2 \cdot 3^{-(n+1)} \sum_{i=0}^n \|P^i f\|_p \leq \|f\|_{5n,p} \quad \text{for all } f \in W^{5n,p}(-1, 0) \text{ and all } n \geq 0.$$

To finish, we use an induction argument over  $n$  and [Lemma 2.5](#) to ensure that

$$\|f\|_{5n,p} \leq (1 + 2K_2)^{n-1} K_2 \sum_{i=0}^n \|P^i f\|_p \quad \text{for all } f \in W^{5n,p}(-1, 0) \text{ and all } n \geq 1. \quad \square$$

**2.3. Smoothing property.** With the previous section in hand, let us show that for any  $u_0 \in L^2(-1, 0)$ , the solution of (1-1) with  $h_1 = h_2 = 0$  is a Gevrey function of order  $\frac{1}{2}$  in the variable  $x$  and  $\frac{5}{2}$  in the variable  $t$ . Precisely, we will prove the existence of positive constants  $M$ ,  $R_1$ , and  $R_2$  such that

$$|\partial_x^p \partial_t^n u(x, t)| \leq \frac{M}{t^{\frac{1}{2}(5n+p+5)}} \frac{p!^{\frac{1}{2}} n!^{\frac{5}{2}}}{R_1^p R_2^n} \quad \text{for all } t \in (0, T] \text{ and all } x \in [-1, 0]. \quad (2-37)$$

To do this, using the inequality (2-37) on intervals of length one, we can assume, without loss of generality, that  $T = 1$ . Consider the operator given by  $Au = -Pu = -\partial_x u - \partial_x^3 u + \partial_x^5 u$ , with  $D(A) = \{u \in H^5(-1, 0); u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\}$ . It has been proven in [Araruna et al. 2012] that  $A$  generates a semigroup of contractions  $\{S(t)\}_{t \geq 0}$  in  $L^2(-1, 0)$ . Moreover, that smoothing effect is verified; that is, for  $u_0 \in L^2(-1, 0)$ , the mild solution  $u(\cdot, t) = S(t)u_0$  of (1-1) with  $h_1 = h_2 = 0$  satisfies  $u \in C([0, 1], L^2(-1, 0)) \cap L^2(0, 1, H^2(-1, 0))$  and

$$\|u\|_{L^2(0, T, H^2(-1, 0))} \leq \sqrt{3} \|u_0\|_{L^2(-1, 0)}. \quad (2-38)$$

From now on, we will for simplicity denote the norm  $\|\cdot\|_{L^2(-1, 0)}$  by  $\|\cdot\|_{L^2}$  and define the spaces

$$\begin{aligned} X_0 &= L^2(-1, 0), \quad X_1 = H_0^1(-1, 0), \quad X_2 = H_0^2(-1, 0), \\ X_3 &= \{u \in H^3(-1, 0); u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\}, \\ X_4 &= \{u \in H^4(-1, 0); u(-1) = u(0) = u_x(-1) = u_x(0) = u_{xx}(0) = 0\}, \\ X_5 &= D(A). \end{aligned}$$

For any  $m \in \{1, 2, 3, 4, 5\}$ ,  $(X_m, \|\cdot\|_{H^m})$  is a Banach space. The next propositions are paramount in our analysis and given several estimates in the  $X_s$ -spaces.

**Proposition 2.7.** *There exists some constant  $C_1 > 0$  such that for all  $u_0 \in L^2(-1, 0)$  and all  $t \in (0, 1]$ ,  $\|S(t)u_0\|_{H^2} \leq (C_1/\sqrt{t})\|u_0\|_{L^2}$ .*

*Proof.* Given  $u_0 \in D(A)$ , semigroup theory ensures that  $S(t)u_0 \in D(A)$  with

$$\frac{d}{dt} S(t)u_0 = AS(t)u_0 = S(t)Au_0 \quad (2-39)$$

and  $u = S(\cdot)u_0 \in C([0, \infty), D(A))$ . From (2-39), we obtain  $\|AS(t)u_0\|_{L^2} = \|S(t)Au_0\|_{L^2}$  and, therefore,  $\|S(t)u_0\|_{D(A)} \leq \|u_0\|_{D(A)}$ .

Using Lemma 2.5 (see also the Remark on page 369) we obtain a positive constant  $C' > 0$  (which does not depend on  $t$ ) such that  $\|S(t)u_0\|_{H^5} \leq C'_1 \|u_0\|_{H^5}$ . The map  $S(t) : (D(A), \|\cdot\|_{H^5}) \rightarrow (D(A), \|\cdot\|_{H^5})$  is continuous. Since  $S(t) : L^2(-1, 0) \rightarrow L^2(-1, 0)$  is also continuous and, by interpolation argument,  $[X_0, X_5]_{2/5} = X_2$ , we have that  $S(t) : (X_2, \|\cdot\|_{H^2}) \rightarrow (X_2, \|\cdot\|_{H^2})$  is continuous with  $\|u(\cdot, t)\|_{H^2} = \|S(t)u_0\|_{H^2} \leq C''_1 \|u_0\|_{H^2}$ , for  $u_0 \in X_2$  and  $t \in (0, 1]$ , where  $C''_1 = \max\{C'_1, 1\}$ .

In this way, given  $u_0 \in X_2$ , for any  $t \in (0, 1]$ , we have

$$\|u(\cdot, t)\|_{H^2}^2 \leq (C''_1)^2 \|u(\cdot, s)\|_{H^2}^2 \quad \text{for all } s \in (0, t].$$

Integrating with respect to  $s$  from 0 to  $t$ , we get

$$t\|u(\cdot, t)\|_{H^2}^2 \leq (C_1'')^2 \int_0^t \|u(\cdot, s)\|_{H^2}^2 ds \leq (C_1'')^2 \|u\|_{L^2(0,1,H^2(-1,0))}^2.$$

Using (2-38), we obtain

$$\|u(\cdot, t)\|_{H^2} \leq \frac{C_1''\sqrt{3}}{\sqrt{t}} \|u_0\|_{L^2}. \quad \square$$

**Proposition 2.8.** *There exists a positive constant  $C_2$  such that, for all  $u_0 \in D(A)$  and all  $t \in (0, 1]$ ,  $\|S(t)u_0\|_{H^7} \leq (C_2/\sqrt{t})\|u_0\|_{H^5}$ .*

*Proof.* First,  $S(t)u_0 \in H^7(-1, 0)$  whenever  $u_0 \in D(A)$ . Indeed, given  $u_0 \in D(A)$ , we have  $u(\cdot, t) = S(t)u_0 \in D(A)$  and (2-39) holds. Then  $Au(\cdot, t) = AS(t)u_0 = S(t)Au_0 \in H^2(-1, 0)$ , which implies that  $\partial_x^5 u(\cdot, t) \in H^2(-1, 0)$  and consequently  $u(\cdot, t) \in H^7(-1, 0)$ . Furthermore, Proposition 2.7 gives

$$\|u(\cdot, t)\|_{H^2} + \|Au(\cdot, t)\|_{H^2} \leq \frac{C_1}{\sqrt{t}} (\|u_0\|_{L^2} + \|Au_0\|_{L^2}). \quad (2-40)$$

Now, let  $v \in H^7(-1, 0)$ . Observe that  $\partial_x^6 v = \partial_x^2 v + \partial_x^4 v - \partial_x P v$  and  $\partial_x^7 v = \partial_x^3 v + \partial_x^5 v - \partial_x^2 P v$ . Then,

$$\|v\|_{H^7} \leq C_2' (\|v\|_{H^5} + \|P v\|_{H^2})$$

for some positive constant  $C_2'$ . Thus, using Lemma 2.5, we obtain

$$\|v\|_{H^7} \leq C_2' (\|v\|_{H^2} + \|A v\|_{H^2}).$$

With this and (2-40), we get that

$$\|u(\cdot, t)\|_{H^7} \leq \frac{C_2' C_1}{\sqrt{t}} (\|u_0\|_{L^2} + \|Au_0\|_{L^2}) = \frac{C_2' C_1}{\sqrt{t}} \|u_0\|_{D(A)} \leq \frac{C_2}{\sqrt{t}} \|u_0\|_{H^5}. \quad \square$$

**Proposition 2.9.** *There exists a constant  $C_3 > 1$  such that, for all  $u_0 \in X_m$  and all  $t \in (0, 1]$ , we have  $\|S(t)u_0\|_{H^{m+2}} \leq (C_3/\sqrt{t})\|u_0\|_{H^m}$ .*

*Proof.* The linear maps  $S(t) : L^2(-1, 0) \rightarrow H^2(-1, 0)$  and  $S(t) : D(A) \rightarrow H^7(-1, 0)$  are continuous by Propositions 2.7 and 2.8. Moreover, there exists  $C_3 > 1$  such that

$$\begin{aligned} \|S(t)u_0\|_{H^2} &\leq \frac{C_3}{\sqrt{t}} \|u_0\|_{L^2} \quad \text{for all } u_0 \in L^2(-1, 0), \\ \|S(t)u_0\|_{H^7} &\leq \frac{C_3}{\sqrt{t}} \|u_0\|_{H^5} \quad \text{for all } u_0 \in D(A). \end{aligned} \quad (2-41)$$

For  $m = 1, 2, 3, 4$ , thanks to the interpolation arguments, it follows that  $S(t)([X_0, X_5]_{\frac{1}{5}m}) \subset [H^2, H^7]_{\frac{1}{5}m}$ , and the maps  $S(t) : [X_0, X_5]_{\frac{1}{5}m} \rightarrow [H^2, H^7]_{\frac{1}{5}m}$  are continuous with

$$\|S(t)u_0\|_{[H^2, H^7]_{\frac{1}{5}m}} \leq \frac{C_3}{\sqrt{t}} \|u_0\|_{[X_0, X_5]_{\frac{1}{5}m}} \quad \text{for all } u_0 \in [X_0, X_5]_{\frac{1}{5}m}.$$

Since  $[H^2, H^7]_{\frac{1}{5}m} = H^{(1-\frac{1}{5}m) \cdot 2 + \frac{1}{2}m \cdot 7}$  and  $[X_0, X_5]_{\frac{1}{5}m} = X_m$ , it follows that the maps  $S(t) : X_m \rightarrow H^{m+2}$  for  $m = 1, 2, 3, 4$  are continuous with

$$\|S(t)u_0\|_{H^{m+2}} \leq \frac{C_3}{\sqrt{t}} \|u_0\|_{H^m} \quad \text{for all } u_0 \in H^m. \quad (2-42)$$

From (2-41) and (2-42) we obtain the desired result.  $\square$

It is convenient to remember that for  $n \in \mathbb{N}$ , we have the inductive definition

$$D(A^n) = \{v \in L^2(-1, 0); v \in D(A^{n-1}) \text{ and } Av \in D(A^{n-1})\}, \quad A^n v = A^{n-1}(Av).$$

So, with this in hand, we have the following result.

**Proposition 2.10.** *Let  $u_0 \in L^2(-1, 0)$  and  $t \in (0, 1]$ . We have for  $u = S(\cdot)u_0$  that:*

(i)  $u(\cdot, t) \in D(A)$  and

$$\|Au(\cdot, t)\|_{L^2} \leq \frac{C_4}{t^{3/2}} \|u_0\|_{L^2}$$

for some positive constant  $C_4 > 1$  (which does not depend on  $t$ ).

(ii)  $u(\cdot, t) \in D(A^n)$  for every  $n \in \mathbb{N}$  and

$$\|A^n u(\cdot, t)\|_{L^2} \leq \frac{C_5^n}{t^{3n/2}} n^{3n/2} \|u_0\|_{L^2},$$

where  $C_5 > \max\{1, C_4\}$  is a constant which does not depend on  $t$ .

(iii)  $u \in C((0, 1], D(A^n))$  for every  $n \in \mathbb{N} \cup \{0\}$ .

*Proof.* (i) Assume  $u_0 \in D(A)$ . Splitting  $[0, t]$  into  $[0, \frac{1}{3}t] \cup [\frac{1}{3}t, \frac{2}{3}t] \cup [\frac{2}{3}t, t]$ , using Proposition 2.9 and taking in mind that  $Au(\cdot, t) = -\partial_x u(\cdot, t) - \partial_x^3 u(\cdot, t) + \partial_x^5 u(\cdot, t)$ , we get that

$$\|Au(\cdot, t)\|_{L^2} \leq \frac{\tilde{C}_4}{t^{3/2}} \|u_0\|_{L^2} \quad \text{for all } u_0 \in D(A), \quad (2-43)$$

where  $\tilde{C}_4 = 3C_3^3/(\sqrt{1/3})^3$  and  $\tilde{C}_4 > 1$ .

Thanks to the inequality (2-43), the linear operator  $S(t) : (D(A), \|\cdot\|_{L^2}) \rightarrow (D(A), \|\cdot\|_{D(A)})$  is bounded, and

$$\|S(t)u_0\|_{D(A)} \leq \frac{2\tilde{C}_4}{t^{3/2}} \|u_0\|_{L^2} \quad \text{for } u_0 \in D(A).$$

Since  $D(A)$  is dense in  $L^2(-1, 0)$ , there exists a bounded linear operator  $\Lambda_t : (L^2(-1, 0), \|\cdot\|_{L^2}) \rightarrow (D(A), \|\cdot\|_{D(A)})$  such that

$$\Lambda_t|_{D(A)} = S(t) \quad \text{and} \quad \|\Lambda_t v\|_{D(A)} \leq \frac{2\tilde{C}_4}{t^{3/2}} \|v\|_{L^2} \quad \text{for all } v \in L^2(-1, 0). \quad (2-44)$$

Given  $u_0 \in L^2(-1, 0)$ , there exists  $(u_k) \subset D(A)$  such that  $u_k \rightarrow u_0$  in  $L^2(-1, 0)$ . Then, from (2-44),

$$\|\Lambda_t u_0 - S(t)u_0\|_{L^2} \leq \frac{2\tilde{C}_4}{t^{3/2}} \|u_0 - u_k\|_{L^2} + \|u_0 - u_k\|_{L^2}.$$

As  $k \rightarrow \infty$ , we obtain  $\Lambda_t u_0 = S(t)u_0$ . Therefore,  $S(t)u_0 \in D(A)$  for  $u_0 \in L^2(-1, 0)$ , and by (2-44),

$$\|Au(\cdot, t)\|_{L^2} = \|AS(t)u_0\|_{L^2} \leq \|S(t)u_0\|_{D(A)} = \|\Lambda_t u_0\|_{D(A)} \leq \frac{C_4}{t^{3/2}} \|u_0\|_{L^2},$$

where  $C_4 = 2\tilde{C}_4$ , proving item (i).

(ii) Assume  $u_0 \in D(A^n)$ . From semigroup theory,  $S(t)u_0 \in D(A^n)$  and  $A^n u(\cdot, t) = AS(t)A^{n-1}u_0$ . Using item (i), we get

$$\|A^n u(\cdot, t)\|_{L^2} = \|AS(t)A^{n-1}u_0\|_{L^2} \leq \frac{C_4}{t^{3/2}} \|A^{n-1}u_0\|_{L^2}. \quad (2-45)$$

Splitting  $[0, t]$  into  $[0, t] = [0, t/n] \cup [t/n, 2t/n] \cup \dots \cup [(n-1)t/n, t]$  and using (2-45) several times, we obtain

$$\begin{aligned} \|A^n u(\cdot, t)\|_{L^2} &\leq \frac{C_4}{t^{3/2}} \|A^{n-1}u(\cdot, (n-1)t/n)\|_{L^2} \\ &\leq \dots \leq \frac{C_4}{t^{3/2}} \frac{C_4}{\left(\frac{n-1}{n}t\right)^{3/2}} \frac{C_4}{\left(\frac{n-2}{n}t\right)^{3/2}} \dots \frac{C_4}{\left(\frac{2}{n}t\right)^{3/2}} \frac{C_4}{\left(\frac{1}{n}t\right)^{3/2}} \|u_0\|_{L^2} \leq \left(\frac{C_4}{\left(\frac{t}{n}\right)^{3/2}}\right)^n \|u_0\|_{L^2}; \end{aligned}$$

thus, for  $u_0 \in D(A^n)$ , it holds that

$$\|A^n u(\cdot, t)\|_{L^2} \leq \frac{C_4^n}{t^{3n/2}} n^{3n/2} \|u_0\|_{L^2}.$$

Now, remark that  $S(t) : (D(A^n), \|\cdot\|_{L^2}) \rightarrow (D(A^n), \|\cdot\|_{D(A^n)})$  is a bounded linear operator, since, for  $u_0 \in D(A^n)$ , the previous estimate ensures that

$$\|S(t)u_0\|_{D(A^n)} = \|S(t)u_0\|_{L^2} + \|A^n S(t)u_0\|_{L^2} \leq \left(1 + \frac{C_4^n}{t^{3n/2}} n^{3n/2}\right) \|u_0\|_{L^2}.$$

Since  $D(A^n)$  is dense in  $L^2(-1, 0)$ , there exists a bounded linear operator  $\Lambda_{t,n} : (L^2(-1, 0), \|\cdot\|_{L^2}) \rightarrow (D(A^n), \|\cdot\|_{D(A^n)})$  such that

$$\Lambda_{t,n}|_{D(A^n)} = S(t) \quad \text{and} \quad \|\Lambda_{t,n}v\|_{D(A^n)} \leq \frac{C_5^n}{t^{3n/2}} n^{3n/2} \|v\|_{L^2} \quad \text{for all } v \in L^2(-1, 0) \quad (2-46)$$

for some constant  $C_5 > \max\{1, C_4\}$  which does not depend on  $t$ .

Given  $u_0 \in L^2(-1, 0)$ , there exists  $(u_k) \subset D(A^n)$  such that  $u_k \rightarrow u_0$  in  $L^2(-1, 0)$ . Thus, from (2-46),

$$\begin{aligned} \|\Lambda_{t,n}u_0 - S(t)u_0\|_{L^2} &\leq \|\Lambda_{t,n}u_0 - \Lambda_{t,n}u_k\|_{D(A^n)} + \|S(t)u_k - S(t)u_0\|_{L^2} \\ &\leq \frac{C_5^n}{t^{3n/2}} n^{3n/2} \|u_0 - u_k\|_{L^2} + \|u_k - u_0\|_{L^2}. \end{aligned}$$

As  $k \rightarrow \infty$ , we obtain  $S(t)u_0 = \Lambda_{t,n}u_0$ . Therefore,  $S(t)u_0 \in D(A)$  for  $u_0 \in L^2(-1, 0)$ , and from (2-46),

$$\|A^n u(\cdot, t)\|_{L^2} \leq \|S(t)u_0\|_{D(A^n)} = \|\Lambda_{t,n}u_0\|_{D(A^n)} \leq \frac{C_5^n}{t^{3n/2}} n^{3n/2} \|u_0\|_{L^2};$$

hence, item (ii) holds.

(iii) Let  $n \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$  be given. By item (ii) we have  $S(t)u_0 \in D(A^n)$ . In particular, this means that  $S(\varepsilon)u_0 \in D(A^n)$  and so, from semigroup theory, we obtain  $S(\cdot)u_0 \in \bigcap_{j=0}^n C^{n-j}([\varepsilon, 1]; D(A^j))$ . Taking  $j = n$ , we get  $S(\cdot)u_0 \in C([\varepsilon, 1]; D(A^n))$ , and since  $\varepsilon \in (0, 1)$  is arbitrary, it follows that  $S(\cdot)u_0 \in C((0, 1]; D(A^n))$ , showing item (iii).  $\square$

The last lemma will be useful to prove the main result of this section.

**Lemma 2.11.** *For every  $n \in \mathbb{N} \cap \{0\}$ , we have  $D(A^n) \subset H^{5n}(-1, 0)$ .*

*Proof.* This is obvious for  $n = 0$  and  $n = 1$ . For  $n = 2$ , given  $v \in D(A^2)$ , there exists  $g \in D(A) \subset H^5$  such that  $Av = g$ , that is,  $\partial_x^5 v = g + \partial_x v + \partial_x^3 v \in H^2$  and therefore  $v \in H^7$ . Thus,  $\partial_x^7 v = \partial_x^2 g + \partial_x^3 v + \partial_x^5 v \in H^2$ , so  $v \in H^9$ . Differentiating again, we get  $\partial_x^9 v = \partial_x^4 g + \partial_x^5 v + \partial_x^7 v \in H^1$ , and so  $v \in H^{10}$ . Therefore  $D(A^2) \subset H^{10}$ .

To conclude the result for any  $n \in \mathbb{N}$ , we proceed by induction. Suppose that for some  $n \geq 1$ , we have  $D(A^n) \subset H^{5n}(-1, 0)$ . Let  $v \in D(A^{n+1})$ , by induction hypothesis, and using the same procedure, we have  $v, Av \in H^{5n}$ , which implies that there exists  $f \in H^{5n}$  with  $Av = f$ , that is,  $\partial_x^5 v \in H^{5n-3}$  and so  $v \in H^{5n+2}$ . Differentiating  $Av = f$  twice, we obtain  $v \in H^{5n+4}$ . Differentiating again twice it follows that  $v \in H^{5n+5}$  and therefore  $D(A^{n+1}) \subset H^{5(n+1)}$ .  $\square$

The previous results ensue directly from the following one.

**Proposition 2.12.** *For any  $u_0 \in L^2(-1, 0)$ , the solution  $u(\cdot, t) = S(t)u_0$  of (1-1) with  $h_1 = h_2 = 0$  satisfies  $u \in C^\infty([-1, 0] \times (0, 1])$ .*

*Proof.* Consider  $n \in \mathbb{N}$ ,  $t \in (0, 1]$  and  $u_0 \in L^2(-1, 0)$ . From Proposition 2.10 we have  $u(\cdot, t) \in D(A^n)$ . Using Lemma 2.11, we obtain  $u(\cdot, t) \in H^{5n}$ . The Sobolev embedding  $H^{5n} \hookrightarrow C^{5n-1}([-1, 0])$  provides  $u(\cdot, t) \in C^{5n-1}([-1, 0])$ . Since  $n \in \mathbb{N}$  is arbitrary, it follows that  $u(\cdot, t) \in C^\infty([-1, 0])$ . On the other hand, given  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , Proposition 2.10 yields that  $u(\cdot, \varepsilon) \in D(A^{n+1})$ , so semigroup theory ensures that  $u \in \bigcap_{j=0}^{n+1} C^{n+1-j}([\varepsilon, 1], D(A^j))$ . In particular, taking  $j = 1$ , we have that  $u \in C^n([\varepsilon, 1], D(A))$ . Using Lemma 2.5 and the embedding  $H^5 \hookrightarrow C^4([-1, 0])$ , we obtain that  $\partial_t^i u(x, \cdot)$  is continuous at  $t_0$  and, as  $t_0 \in [\varepsilon, 1]$  is arbitrary, we conclude that  $\partial_t^i u(x, \cdot)$  is continuous for  $i = 0, 1, \dots, n$ . Since  $n \in \mathbb{N}$  and  $x \in [-1, 0]$  are arbitrary, we get  $u(x, \cdot) \in C^\infty([\varepsilon, 1])$ . Furthermore,  $u(x, \cdot) \in C^\infty((0, 1])$ , and we get the result.  $\square$

We are now in a position to prove a smoothness property for the solutions of (1-1). With these auxiliary results in hand, the main result of this section is the following one.

**Proposition 2.13.** *Let  $u_0 \in L^2(-1, 0)$  and  $h_1(t) = h_2(t) = 0$  for  $t \in [0, 1]$ . Then the corresponding solution  $u$  of (1-1) satisfies  $u \in G^{\frac{1}{2}, \frac{5}{2}}([-1, 0] \times [\varepsilon, 1])$ , for all  $\varepsilon \in (0, 1)$ ; that is, we can find positive constants  $M, R_1, R_2$  such that*

$$|\partial_x^p \partial_t^n u(x, t)| \leq \frac{M}{t^{\frac{1}{2}(5n+p+5)}} \frac{p!^{\frac{1}{2}} n!^{\frac{5}{2}}}{R_1^p R_2^n}.$$

*Proof.* Consider  $p \in \mathbb{N} \cup \{0\}$  and choose  $n \in \mathbb{N}$  such that  $5n - 5 \leq p \leq 5n - 1$ . Then the Sobolev embedding, [Lemma 2.6](#) and [Proposition 2.10](#) ensure that

$$\begin{aligned} & \|\partial_x^p u(\cdot, t)\|_\infty \\ & \leq C_6 K_3^n \sum_{i=0}^n \|P^i u(\cdot, t)\|_{L^2} = C_6 K_3^n \left( \|u(\cdot, t)\|_{L^2} + \sum_{i=1}^n \|A^i u(\cdot, t)\|_{L^2} \right) \leq C_6 K_3^n \left( 1 + \sum_{i=1}^n \frac{C_5^i}{t^{\frac{3}{2}i}} i^{\frac{3}{2}i} \right) \|u_0\|_{L^2} \\ & \leq C_6 K_3^n (n+1) \frac{C_5^n}{t^{\frac{3}{2}n}} n^{\frac{3}{2}n} \|u_0\|_{L^2} = C_6 K_3^n (n+1) \frac{C_5^n}{t^{\frac{5}{2}n}} \frac{(5n)^{\frac{5}{2}n}}{5^{\frac{5}{2}n}} \|u_0\|_{L^2}. \quad (2-47) \end{aligned}$$

Using Stirling's formula, we get

$$(5n)^{5n} \sim \frac{e^{5n} (5n)!}{(2\pi)^{\frac{1}{2}} (5n)^{\frac{1}{2}}} \iff (5n)^{\frac{5}{2}n} \sim \frac{e^{\frac{5}{2}n} (5n)!^{\frac{1}{2}}}{(2\pi)^{\frac{1}{4}} (5n)^{\frac{1}{4}}} \iff \frac{(5n)^{\frac{5}{2}n}}{5^{\frac{5}{2}n}} \sim \frac{1}{(2\pi)^{\frac{1}{4}} (5n)^{\frac{1}{4}}} \left(\frac{1}{5}e\right)^{\frac{5}{2}n} (5n)!^{\frac{1}{2}},$$

which implies that

$$\frac{(5n)^{\frac{5}{2}n}}{5^{\frac{5}{2}n}} \leq \frac{C_7}{(2\pi)^{\frac{1}{4}} (5n)^{\frac{1}{4}}} \left(\frac{1}{5}e\right)^{\frac{5}{2}n} (5n)!^{\frac{1}{2}} \leq C_7 \left(\frac{1}{5}e\right)^{\frac{5}{2}n} (5n)!^{\frac{1}{2}}$$

for some constant  $C_7 > 0$ . Since  $(n+1) \leq e^{\frac{5}{2}n}$  for  $n \in \mathbb{N}$ , the inequality (2-47) gives us

$$\|\partial_x^p u(\cdot, t)\|_\infty \leq C_6 K_3^n e^{\frac{5}{2}n} \frac{C_5^n}{t^{\frac{5}{2}n}} C_7 \left(\frac{1}{5}e\right)^{\frac{5}{2}n} (5n)!^{\frac{1}{2}} \|u_0\|_{L^2} = C_6 C_7 \frac{C_5^n}{t^{\frac{5}{2}n}} K_3^n \left(\frac{1}{5}e^2\right)^{\frac{5}{2}n} (5n)!^{\frac{1}{2}} \|u_0\|_{L^2}.$$

Remember that  $5n - 5 \leq p \leq 5n - 1$ , which allows us to write  $p = 5n - r$  with  $r \in \{1, 2, 3, 4, 5\}$ ; that is,  $5n = p + r$  with  $r \in \{1, 2, 3, 4, 5\}$ . Using (2-3), it follows that

$$\begin{aligned} \|\partial_x^p u(\cdot, t)\|_\infty & \leq C_6 C_7 \frac{C_5^{\frac{1}{5}(p+r)}}{t^{\frac{1}{2}(p+r)}} K_3^{\frac{1}{5}(p+r)} \left(\frac{1}{5}e^2\right)^{\frac{1}{2}(p+r)} (p+r)!^{\frac{1}{2}} \|u_0\|_{L^2} \\ & \leq C_6 C_7 C_5^{\frac{1}{5}r} K_3^{\frac{1}{5}r} \left(\frac{1}{5}e^2\right)^{\frac{1}{2}r} \frac{C_5^{\frac{1}{5}p}}{t^{\frac{1}{2}(p+r)}} K_3^{\frac{1}{5}p} \left(\frac{1}{5}e^2\right)^{\frac{1}{2}p} (2^p 2^r p! r!)^{\frac{1}{2}} \|u_0\|_{L^2} \\ & = C_6 C_7 C_5^{\frac{1}{5}r} K_3^{\frac{1}{5}r} \left(\frac{1}{5}e^2\right)^{\frac{1}{2}r} 2^{\frac{1}{2}r} r!^{\frac{1}{2}} \|u_0\|_{L^2} \frac{1}{t^{\frac{1}{2}(p+r)}} \left(\frac{C_5^{\frac{1}{5}} K_3^{\frac{1}{5}} e 2^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)^p p!^{\frac{1}{2}}. \end{aligned}$$

Consequently,

$$\|\partial_x^p u(\cdot, t)\|_\infty \leq \frac{C_8}{t^{\frac{1}{2}(p+r)}} \left(\frac{C_5^{\frac{1}{5}} K_0^{\frac{1}{5}} e 2^{\frac{1}{2}}}{5^{\frac{1}{2}}}\right)^p p!^{\frac{1}{2}} = \frac{C_8}{t^{\frac{1}{2}(p+r)}} \frac{p!^{\frac{1}{2}}}{(5^{\frac{1}{2}} C_5^{-\frac{1}{5}} K_0^{-\frac{1}{5}} e^{-1} 2^{-\frac{1}{2}})^p},$$

with

$$C_8 = C_8(r) := C_6 C_7 C_5^{\frac{1}{5}r} K_3^{\frac{1}{5}r} \left(\frac{1}{5}e^2\right)^{\frac{1}{2}r} 2^{\frac{1}{2}r} r!^{\frac{1}{2}} \|u_0\|_{L^2} \quad \text{and} \quad K_0 = K_3 + 1.$$

Defining  $R = 5^{\frac{1}{2}} C_5^{-\frac{1}{5}} K_0^{-\frac{1}{5}} e^{-1} 2^{-\frac{1}{2}}$ , it follows that  $R \in (0, 1)$  and

$$\|\partial_x^p u(\cdot, t)\|_\infty \leq \frac{C_8}{t^{\frac{1}{2}(p+r)}} \frac{p!^{\frac{1}{2}}}{R^p}.$$

Here,  $p \geq 0$ ,  $r \in \{1, 2, 3, 4, 5\}$  and  $t \in (0, 1]$ . Observe that, as  $t \in (0, 1]$  and  $0 \leq r \leq 5$ ,

$$\|\partial_x^p u(\cdot, t)\|_\infty \leq \frac{C_8}{t^{\frac{1}{2}(p+5)}} \frac{p!^{\frac{1}{2}}}{R^p} \quad \text{for all } p \geq 0 \text{ and all } t \in (0, 1]. \quad (2-48)$$

Finally, for every  $n, p \geq 0$  from (1-4), we have

$$\partial_t^n \partial_x^p u = \partial_x^p \partial_t^n u = \partial_x^p (-(-1)^{n-1} P^n u) = \partial_x^p ((-1)^n P^n u) = (-1)^n P^n \partial_x^p u.$$

Using Newton's binomial theorem, we have that

$$P^n \partial_x^p u = \sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j} (-\partial_x^5)^j (\partial_x^3)^{q-j} \partial_x^{n-q} \partial_x^p u = \sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j} (-1)^j \partial_x^{5j} \partial_x^{3q-3j} \partial_x^{n-q} \partial_x^p u.$$

Thus, from (2-48), for  $(x, t) \in [-1, 0] \times (0, 1]$ ,

$$\begin{aligned} |\partial_t^n \partial_x^p u(x, t)| &\leq \sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j} |\partial_x^{n+2q+2j+p} u(x, t)| \\ &\leq \sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j} \frac{C_8}{t^{\frac{1}{2}(n+2q+2j+p+5)}} \frac{(n+2q+2j+p)!^{\frac{1}{2}}}{R^{n+2q+2j+p}}. \end{aligned}$$

Since  $t \in (0, 1]$ ,  $R < 1$  and  $n+2q+2j+p \leq 5n+p$ , it follows that

$$|\partial_t^n \partial_x^p u(x, t)| \leq \frac{C_8}{t^{\frac{1}{2}(5n+p+5)}} \frac{(5n+p)!^{\frac{1}{2}}}{R^{5n+p}} \sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j}.$$

Noting that

$$2^q = (1+1)^q = \sum_{j=0}^q \binom{q}{j} 1^j \cdot 1^{q-j} = \sum_{j=0}^q \binom{q}{j}$$

and

$$\sum_{q=0}^n \binom{n}{q} \sum_{j=0}^q \binom{q}{j} = \sum_{q=0}^n \binom{n}{q} 2^q = \sum_{q=0}^n \binom{n}{q} 2^q \cdot 1^{n-q} = (2+1)^n = 3^n,$$

using (2-3) one more time yields

$$|\partial_t^n \partial_x^p u(x, t)| \leq \frac{C_8}{t^{\frac{1}{2}(5n+p+5)}} \frac{3^n \cdot 2^{\frac{5}{2}n} 2^{\frac{1}{2}p} (5n)!^{\frac{1}{2}} p!^{\frac{1}{2}}}{R^{5n} R^p}.$$

Stirling's formula gives us

$$(5n)! \sim \left(\frac{5n}{e}\right)^{5n} \sqrt{2\pi 5n} = (10\pi)^{\frac{1}{2}} \left(\frac{5n}{e}\right)^{5n} n^{\frac{1}{2}}. \quad (2-49)$$

We also have  $n! \sim (n/e)^n \sqrt{2\pi n} = (2\pi)^{1/2} (n/e)^n n^{1/2}$ , hence

$$\frac{5^{5n} n!^5}{(2\pi)^{\frac{5}{2}n} n^{\frac{5}{2}}} \sim \left(\frac{5n}{e}\right)^{5n}.$$

This previous relation together with (2-49) leads to

$$(5n)! \sim (10\pi)^{\frac{1}{2}} \frac{5^{5n} n!^5}{(2\pi)^{\frac{5}{2}} n^{\frac{5}{2}}} n^{\frac{1}{2}} = \frac{(10\pi)^{\frac{1}{2}} 5^{5n} n!^5}{(2\pi)^{\frac{5}{2}} n^2} \sim \frac{(10\pi)^{\frac{1}{4}} 5^{\frac{5}{2}n} n!^{\frac{5}{2}}}{(2\pi)^{\frac{5}{4}} n}.$$

Thus, there exists a constant  $C_9 > 0$  such that  $(5n)!^{\frac{1}{2}} \leq C_9 5^{\frac{5}{2}n} n!^{\frac{5}{2}}$ . Hence,

$$|\partial_t^n \partial_x^p u(x, t)| \leq \frac{C_8 C_9}{t^{\frac{1}{2}(5n+p+5)}} \left( \frac{3 \cdot 2^{\frac{5}{2}} \cdot 5^{\frac{5}{2}}}{R^5} \right)^n n!^{\frac{5}{2}} \left( \frac{2^{\frac{1}{2}}}{R} \right)^p p!^{\frac{1}{2}} = \frac{C_8 C_9}{t^{\frac{1}{2}(5n+p+5)}} \frac{n!^{\frac{5}{2}}}{(3^{-1} \cdot 10^{-\frac{5}{2}} \cdot R^5)^n} \frac{p!^{\frac{1}{2}}}{(2^{-\frac{1}{2}} R)^p},$$

and the result is achieved with  $M = C_8 C_9$ ,  $R_1 = 2^{-\frac{1}{2}} R$  and  $R_2 = 3^{-1} \cdot 10^{-\frac{5}{2}} \cdot R^5$ .  $\square$

**2.4. Null controllability results.** Let us now prove the null controllability result. Precisely, employing two flat output controls, the solution of (1-1) satisfies  $u(\cdot, T) = 0$ .

*Proof of Theorem 1.1.* Consider  $u_0 \in L^2(-1, 0)$  and denote by  $\bar{u}$  the solution of (1-1) for  $h_1 = h_2 = 0$ . From Proposition 2.13 we have, for  $\varepsilon \in (0, T)$ , that  $\bar{u} \in G^{\frac{1}{2}, \frac{5}{2}}([-1, 0] \times [\varepsilon, T])$ . In particular,  $\partial_x^3 \bar{u}(0, t)$ ,  $\partial_x^4 \bar{u}(0, t) \in G^{\frac{5}{2}}([\varepsilon, T])$  for any  $\varepsilon \in (0, T)$ . Choose  $\tau \in (0, T)$  and define

$$y(t) = \phi_s \left( \frac{t - \tau}{T - \tau} \right) \partial_x^3 \bar{u}(0, t) \quad \text{and} \quad z(t) = \phi_s \left( \frac{t - \tau}{T - \tau} \right) \partial_x^4 \bar{u}(0, t),$$

where  $\phi_s$  is the function given by

$$\phi_s(r) = \begin{cases} 1 & \text{if } r \leq 0, \\ 0 & \text{if } r \geq 1, \\ e^{-K/(1-r)^\sigma} / (e^{-K/r^\sigma} + e^{-K/(1-r)^\sigma}) & \text{if } r \in (0, 1), \end{cases}$$

with  $K > 0$  and  $\sigma := (s - 1)^{-1}$ . As  $\phi_s$  is Gevrey of order  $s$  (see, for example, [Martin et al. 2018]) and  $s \geq \frac{5}{2}$ , we infer that  $y, z \in G^s([\varepsilon, T])$  for all  $\varepsilon \in (0, T)$ . Then, defining  $u : [-1, 0] \times (0, T] \rightarrow \mathbb{R}$  by

$$u(x, t) = \begin{cases} u_0(x) & \text{if } t = 0, \\ \sum_{i \geq 0} f_i(x) y^{(i)}(t) + \sum_{i \geq 0} g_i(x) z^{(i)}(t) & \text{if } t \in (0, T], \end{cases}$$

Proposition 2.3 gives us that  $u$  satisfies (2-11) with  $u \in G^{\frac{1}{s}, s}([-1, 0] \times [\varepsilon, T])$  for all  $\varepsilon \in (0, T)$ . In particular,

$$\partial_x^m u(0, t) = 0 \quad \text{for } m = 0, 1, 2, \quad \partial_x^3 u(0, t) = y(t), \quad \text{and} \quad \partial_x^4 u(0, t) = z(t).$$

Furthermore, by construction,

$$\partial_x^m \bar{u}(0, t) = 0, \quad \text{for } m = 0, 1, 2$$

and for  $t \in (0, \tau)$ ,

$$y(t) = \underbrace{\phi_s \left( \frac{t - \tau}{T - \tau} \right)}_{=1} \partial_x^3 \bar{u}(0, t) = \partial_x^3 \bar{u}(0, t) \quad \text{and} \quad z(t) = \underbrace{\phi_s \left( \frac{t - \tau}{T - \tau} \right)}_{=1} \partial_x^4 \bar{u}(0, t) = \partial_x^4 \bar{u}(0, t).$$

Therefore  $\partial_x^m u(0, t) = \partial_x^m \bar{u}(0, t)$  for  $m = 0, 1, 2, 3, 4$  and  $t \in (0, \tau)$ . Thanks to Holmgren's theorem, we conclude that  $u(x, t) = \bar{u}(x, t)$  for all  $(x, t) \in [-1, 0] \times (0, \tau)$ . Hence,  $u \in C([0, T], L^2(-1, 0))$  and it solves the system (1-1) with

$$h_1(t) = \sum_{i \geq 0} f_i(-1)y^{(i)}(t) + \sum_{i \geq 0} g_i(-1)z^{(i)}(t) \quad \text{and} \quad h_2(t) = \sum_{i \geq 0} f_{ix}(-1)y^{(i)}(t) + \sum_{i \geq 0} g_{ix}(-1)z^{(i)}(t).$$

Observe that  $h_1, h_2 \in G^s([0, T])$  and  $h_1(t) = h_2(t) = 0$  for  $0 < t < \tau$  since, for all  $t \in (0, \tau)$ ,

$$h_1(t) = u(-1, t) = \bar{u}(-1, t) = 0, \quad h_2(t) = u_x(-1, t) = \bar{u}_x(-1, t) = 0.$$

Finally, once we have  $\text{supp } y^{(i)} \subset \text{supp } y \subset (-\infty, T)$  and  $\text{supp } z^{(i)} \subset \text{supp } z \subset (-\infty, T)$ , we get that  $y^{(i)}(T) = 0$  and  $z^{(i)}(T) = 0$  for every  $i \geq 0$ , so that  $u(\cdot, T) = 0$  and the first main result is proved.  $\square$

### 3. A class of reachable functions

We will now establish a class of sets that can be reachable from 0 by the system (1-1). Our goal is to prove that, given  $u_1 \in \mathcal{R}_R$  (see the definition in (1-5)), one can find control inputs  $h_1$  and  $h_2$  for which the solution of (1-1), with  $u_0 = 0$ , satisfies  $u(x, T) = u_1(x)$ .

**3.1. Auxiliary results.** The following auxiliary results will be necessary in our proofs. The first establishes the flatness property for the limit case  $s = 5$ .

**Proposition 3.1.** *Let  $R > 2^{\frac{1}{5}}$  and  $y, z \in G^5([0, T])$  with*

$$|y^{(j)}(t)|, |z^{(j)}(t)| \leq M \frac{(5j)!}{R^{5j}} \quad \text{for all } j \geq 0 \text{ and all } t \in [0, T]. \quad (3-1)$$

*Then, defining  $u(x, t)$  as in (2-4), we have that  $u \in G^{1,5}([-1, 0] \times [0, T])$  solves (2-11).*

*Proof.* Let  $m, k \geq 0$ . For  $j \geq k$  we have, from (3-1) and Lemma 2.2, that

$$\sum_{j=0}^{\infty} |f_{j-k}(x)y^{(j+m)}(t)| \leq \sum_{j \geq k} 2^{j-k} \frac{|x|^{5(j-k)+1}}{[5(j-k)+1]!} M \frac{[5(j+m)]!}{R^{5(j+m)}}.$$

Pick  $l = 5(j-k)$  and  $N = 5(k+m)$ , so  $l+N = 5(j+m)$  gives that

$$\sum_{j=0}^{\infty} |f_{j-k}(x)y^{(j+m)}(t)| \leq M \sum_{l \geq 0} 2^{\frac{l}{5}} \frac{1}{(l+1)!} \frac{(l+N)!}{R^{l+N}}.$$

If  $N \leq 1$ , then

$$\sum_{j=0}^{\infty} |f_{j-k}(x)y^{(j+m)}(t)| \leq \frac{M}{R^N} \sum_{l \geq 0} \left( \frac{1}{R \cdot 2^{-\frac{1}{5}}} \right)^l < \infty.$$

Assume  $N > 1$  and note that  $(l + N)! = (l + N)(l + N - 1) \cdots (l + 2)(l + 1)!$ , so that

$$\begin{aligned} \sum_{j=0}^{\infty} |f_{j-k}(x) y^{(j+m)}(t)| &\leq M \sum_{l \geq 0} 2^{\frac{1}{5}l} \frac{(l + N)(l + N - 1) \cdots (l + 2)}{R^{l+N}} \\ &\leq M \sum_{q \geq 0} \sum_{qN \leq l < (q+1)N} \frac{2^{\frac{1}{5}(l+N)} (l + N)^{N-1}}{R^{l+N}} \\ &= M \sum_{q \geq 0} \sum_{qN \leq l < (q+1)N} \frac{[(q + 2)N]^{N-1}}{(2^{-\frac{1}{5}} R)^{l+N}}. \end{aligned}$$

For  $l > qN$ , we have  $l + N \geq (q + 1)N$  and, since  $2^{-\frac{1}{5}} R > 1$ , this ensures that

$$\frac{1}{(2^{-\frac{1}{5}} R)^{l+N}} \leq \frac{1}{(2^{-\frac{1}{5}} R)^{(q+1)N}}.$$

Thus, thanks to the relation (2-21), the estimate

$$\sum_{j=0}^{\infty} |\partial_t^m P^k(f_j(x) y^{(j)}(t))| \leq M \sum_{q \geq 0} \sum_{qN \leq l < (q+1)N} \frac{(q + 2)^{N-1} N^{N-1}}{(2^{-\frac{1}{5}} R)^{(q+1)N}} \leq M N^N \sum_{q \geq 0} \left( \frac{q + 2}{\tilde{R}^{q+1}} \right)^N$$

is verified with  $\tilde{R} = 2^{-\frac{1}{5}} R$ . Pick any  $\sigma \in (0, 1)$  and define  $f_{\sigma}(x) = (x + 2)/(\tilde{R}^{1-\sigma})^{x+1}$ . Note that  $\lim_{x \rightarrow \infty} (\tilde{R}^{1-\sigma})^{x+1} = \infty$ , since  $\tilde{R} > 1$ . L'Hospital's rule ensures that

$$\lim_{x \rightarrow \infty} f_{\sigma}(x) = \lim_{x \rightarrow \infty} \frac{1}{\ln(\tilde{R}^{1-\sigma})(\tilde{R}^{1-\sigma})^{x+1}} = 0,$$

and so

$$\frac{q + 2}{(\tilde{R}^{1-\sigma})^{q+1}} \rightarrow 0 \quad \text{as } q \rightarrow \infty.$$

Defining  $a := \sup_{q \geq 0} (q + 2)/(\tilde{R}^{1-\sigma})^{q+1}$ , it holds that

$$\sum_{q \geq 0} \left( \frac{q + 2}{\tilde{R}^{q+1}} \right)^N \leq \frac{a^N}{\tilde{R}^{N\sigma}} \sum_{q \geq 0} \frac{1}{\tilde{R}^{N\sigma q}} = \frac{a^N}{\tilde{R}^{N\sigma}} \left( \frac{1}{1 - 1/\tilde{R}^{N\sigma}} \right).$$

Once we have the convergence

$$\alpha_N := \frac{1}{1 - 1/\tilde{R}^{N\sigma}} \rightarrow 1 \quad \text{as } N \rightarrow \infty,$$

we can define  $\tilde{M} := \sup_{N > 1} \alpha_N$  to get

$$\sum_{q \geq 0} \left( \frac{q + 2}{\tilde{R}^{q+1}} \right)^N \leq \tilde{M} \frac{a^N}{\tilde{R}^{N\sigma}}.$$

Consequently,

$$\sum_{j=0}^{\infty} |\partial_t^m P^k(f_j(x) y^{(j)}(t))| \leq M N^N \tilde{M} \frac{a^N}{\tilde{R}^{N\sigma}},$$

and using Stirling's formula, we get that

$$\frac{a^N N^N}{\tilde{R}^{N\sigma}} \sim \frac{1}{\sqrt{2\pi}} \frac{a^N e^N}{\tilde{R}^{N\sigma}} \frac{N!}{N^{\frac{1}{2}}},$$

which ensures the estimate

$$\sum_{j=0}^{\infty} |\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq M' \left( \frac{ae}{\tilde{R}^\sigma} \right)^N \frac{N!}{N^{\frac{1}{2}}},$$

for some constant  $M' > 0$ . Noting that  $N! = (5k + 5m)! \leq 2^{5k} 2^{5m} (5k)! (5m)!$  and using Stirling's formula again, namely,  $(5m)! \sim 5^{5m+\frac{1}{2}} (\sqrt{2\pi m})^{-4} m!^5$ , we get

$$\sum_{j=0}^{\infty} |\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq \sqrt{5} M'' \left( \frac{2ae}{\tilde{R}^\sigma} \right)^{5k} (5k)! \left( \frac{10ae}{\tilde{R}^\sigma} \right)^{5m} m!^5 \frac{1}{(k+1)^{\frac{1}{2}}}.$$

Now, define  $R_1 = (2ae)^{-1} \tilde{R}^\sigma$ ,  $R_2 = [(10ae)^{-1} \tilde{R}]^5$  and  $M''' = \sqrt{5} M''$ . It follows that

$$\sum_{j=0}^{\infty} |\partial_t^m P^k(f_j(x)y^{(j)}(t))| \leq M''' \frac{(5k)!}{R_1^{5k}} \frac{m!^5}{R_2^m} \frac{1}{(k+1)^{\frac{1}{2}}}.$$

Observe that we can assume  $R_1 < 1$ . Let  $K_3 > 0$  as in [Lemma 2.6](#) for  $p = \infty$ . Then, the previous inequality yields that

$$\sum_{j=0}^{\infty} \|\partial_t^m(f_j y^{(j)}(t))\|_{5i,\infty} \leq M''' K_3^i \frac{m!^5 (5i)!}{R_2^m R_1^{5i}} \sum_{k=0}^i \frac{1}{(k+1)^{\frac{1}{2}}}$$

for all  $i \geq 0$ . Given  $m, n \geq 0$ , consider  $i \geq 0$  such that  $n \in \{5i - r; r = 0, 1, 2, 3, 4\}$ . Thus,

$$\sum_{j=0}^{\infty} |\partial_x^n \partial_t^m(f_j(x)y^{(j)}(t))| \leq M''' K_3^i \frac{m!^5 (5i)!}{R_2^m R_1^{5i}} \sum_{k=0}^i \frac{1}{(k+1)^{\frac{1}{2}}}$$

for  $(x, t) \in [-1, 0] \times [0, T]$ . Analogously, one can see that

$$\sum_{j=0}^{\infty} |\partial_x^n \partial_t^m(g_j(x)z^{(j)}(t))| \leq M''' K_3^i \frac{m!^5 (5i)!}{R_2^m R_1^{5i}} \sum_{k=0}^i \frac{1}{(k+1)^{\frac{1}{2}}}.$$

Therefore, these series are uniformly convergent on  $[-1, 0] \times [0, T]$  for all  $m, n \geq 0$ , so that the function  $u$  defined by (2-4) satisfies  $u \in C^\infty([-1, 0] \times [0, T])$ . Furthermore,

$$\sum_{j=0}^{\infty} |\partial_x^n \partial_t^m(f_j(x)y^{(j)}(t))| + \sum_{j=0}^{\infty} |\partial_x^n \partial_t^m(g_j(x)z^{(j)}(t))| \leq M''' K_3^i \frac{m!^5 (5i)!}{R_2^m R_1^{5i}} \sum_{k=0}^i \frac{1}{(k+1)^{\frac{1}{2}}}.$$

On the other hand, given  $\rho > 1$ , one can get a constant  $c > 0$  such that

$$\sum_{k=0}^i \frac{1}{(k+1)^{\frac{1}{2}}} \leq i + 1 \leq c\rho^i.$$

Hence, adjusting the constants  $M'''$  and  $K_3$ , we can write

$$|\partial_x^n \partial_t^m u(x, t)| \leq M''' K_3^i \frac{m!^5 (5i)!}{R_2^m R_1^{5i}}.$$

Since  $n = 5i - r$  with  $r \in \{0, 1, 2, 3, 4\}$ , we have

$$\frac{K_3^i}{R_1^{5i}} (5i)! \leq \frac{K_3^{\frac{1}{5}r} \cdot 2^r \cdot r!}{R_1^r} \cdot \frac{n!}{(K_3^{-\frac{1}{5}} \cdot 2^{-1} \cdot R_1)^n}.$$

Defining

$$\hat{M} = 2M''' \left( \max_{0 \leq r \leq 4} \frac{K_3^{\frac{1}{5}r} \cdot 2^r \cdot r!}{R_1^r} \right) \quad \text{and} \quad R'_1 = K_3^{-\frac{1}{5}} \cdot 2^{-1} \cdot R_1,$$

it follows that

$$|\partial_x^n \partial_t^m u(x, t)| \leq \hat{M} \frac{n!}{(R'_1)^n} \frac{m!^5}{R_2^m} \quad \text{for all } n, m \geq 0 \text{ and all } (x, t) \in [-1, 0] \times [0, T]. \quad \square$$

The next result is a particular case of [Martin et al. 2016b, Proposition 3.6] with  $a_0 = 1$  and, for  $p \geq 1$ ,  $a_p = [5p(5p-1)(5p-2)(5p-3)(5p-4)]^{-1}$ .

**Proposition 3.2.** *Let  $(d_q)_{q \geq 0}$  be a sequence of real numbers satisfying  $|d_q| \leq CH^q (5q)!$ , for all  $q \geq 0$  and for some constants  $H > 0$  and  $C > 0$ . Then, for each  $\tilde{H} > e^{e^{-1}} H$ , one can find a function  $f \in C^\infty(\mathbb{R})$  such that  $f^{(q)}(0) = d_q$  for all  $q \geq 0$ , and*

$$|f^{(q)}(x)| \leq C \tilde{H}^q (5q)! \quad \text{for all } q \geq 0 \text{ and all } x \in \mathbb{R}.$$

The next lemma is a consequence of the theory of analytic functions.

**Lemma 3.3.** *Let  $\psi \in G^1([-1, 0])$  be such that  $\partial_x^j P^n \psi(0) = 0$  for all  $n \geq 0$  and all  $j = 0, 1, 2, 3, 4$ . Then  $\psi \equiv 0$ .*

*Proof.* First, remember (from the proof of Proposition 2.13) that, for all  $n \geq 0$ ,

$$P^n = \sum_{q=0}^n \binom{n}{q} \sum_{k=0}^q \binom{q}{k} (-1)^k \partial_x^{n+2q+2k} \implies \partial_x^j P^n = \sum_{q=0}^n \binom{n}{q} \sum_{k=0}^q \binom{q}{k} (-1)^k \partial_x^{n+2q+2k+j} \quad \text{for all } j \geq 0.$$

We claim that for any  $n \geq 0$ ,

$$\partial_x^j \psi(0) = 0 \quad \text{for all } j \in \{0, 1, \dots, 5n+4\}. \quad (3-2)$$

To prove this, we use induction on  $n$ . For  $n = 0$ , this immediately follows from the hypothesis, since

$$\partial_x^j \psi(0) = \partial_x^j P^0 \psi(0) = 0, \quad j = 0, 1, 2, 3, 4. \quad (3-3)$$

Let us also analyze the case  $n = 1$ . Using the hypothesis and (3-3), we obtain

$$\begin{aligned} P\psi(0) = 0 &\implies \partial_x^5 \psi(0) = 0, \quad \partial_x P\psi(0) = 0 \implies \partial_x^6 \psi(0) = 0, \quad \partial_x^2 P\psi(0) = 0 \implies \partial_x^7 \psi(0) = 0, \\ \partial_x^3 P\psi(0) = 0 &\implies \partial_x^8 \psi(0) = 0, \quad \partial_x^4 P\psi(0) = 0 \implies \partial_x^9 \psi(0) = 0. \end{aligned}$$

Combining this with (3-3), we get (3-2) for  $n = 1$ . Now, suppose that (3-2) holds for some  $n \geq 1$ . We will show that

$$\partial_x^j \psi(0) = 0 \quad \text{for all } j \in \{0, 1, \dots, 5(n+1) + 4\}.$$

By the induction hypothesis, it is sufficient to show that  $\partial_x^j \psi(0) = 0$  for  $j = 5(n+1) + r$  with  $r = 0, 1, 2, 3, 4$ . From the hypothesis,  $P^{n+1} \psi(0) = 0$ , that is,

$$\sum_{q=0}^{n+1} \binom{n+1}{q} \sum_{k=0}^q \binom{q}{k} (-1)^k \partial_x^{n+1+2q+2k} \psi(0) = 0.$$

Thus,

$$\sum_{q=0}^n \binom{n+1}{q} \sum_{k=0}^q \binom{q}{k} (-1)^k \partial_x^{n+1+2q+2k} \psi(0) + \binom{n+1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \partial_x^{n+1+2(n+1)+2k} \psi(0) = 0.$$

Note that  $n+1+2q+2k \leq 5n+1$  for  $0 \leq k \leq q \leq n$ . From the induction hypothesis, it follows that  $\partial_x^{n+1+2q+2k} \psi(0) = 0$  and the last equality becomes

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \partial_x^{3(n+1)+2k} \psi(0) = 0,$$

or, equivalently,

$$\sum_{k=0}^n \binom{n+1}{k} (-1)^k \partial_x^{3(n+1)+2k} \psi(0) + \binom{n+1}{n+1} (-1)^{n+1} \partial_x^{5(n+1)} \psi(0) = 0.$$

For  $0 \leq k \leq n$ , we have  $3(n+1)+2k \leq 5n+3$ , and the induction hypothesis gives us  $\partial_x^{3(n+1)+2k} \psi(0) = 0$  and therefore, from the last equality, we get that  $\partial_x^{5(n+1)} \psi(0) = 0$ . In a similar way,

$$\partial_x^r P^{n+1} \psi(0) = 0 \implies \partial_x^{5(n+1)+r} \psi(0) = 0, \quad r = 1, 2, 3, 4,$$

finishing the proof of (3-2), and we conclude that  $\partial_x^j \psi(0) = 0$ , for all  $j \geq 0$ . Since  $\psi$  is analytic in  $[-1, 0]$ , it follows that  $\psi \equiv 0$ .  $\square$

**3.2. Reachable states.** We are now in a position to prove the second main result of the article.

*Proof of Theorem 1.2.* Let  $R > 2R_0$  and  $u_1 \in \mathcal{R}_R$ . We will later show that  $u_1$  can be written in the form

$$u_1(x) = \sum_{i \geq 0} c_i f_i(x) + \sum_{i \geq 0} b_i g_i(x) \quad \text{for all } x \in [-1, 0]. \quad (3-4)$$

Assume for a moment that (3-4) holds with convergence in  $W^{n,\infty}(-1, 0)$  for all  $n \geq 0$ . Then, using (2-20) and (2-24), we obtain

$$P^n u_1(x) = (-1)^n \sum_{i \geq n} c_i f_{i-n}(x) + (-1)^n \sum_{i \geq n} b_i g_{i-n}(x)$$

and

$$\partial_x^j P^n u_1(x) = (-1)^n \sum_{i \geq n} c_i \partial_x^j f_{i-n}(x) + (-1)^n \sum_{i \geq n} b_i \partial_x^j g_{i-n}(x) \quad \text{for all } j \geq 0.$$

From (2-5)–(2-8), it follows that

$$\partial_x^3 P^n u_1(0) = (-1)^n c_n \partial_x^3 f_0(0) + (-1)^n \sum_{i > n} c_i \partial_x^3 f_{i-n}(0) + (-1)^n \sum_{i \geq n} b_i \partial_x^3 g_{i-n}(0) = (-1)^n c_n$$

and

$$\partial_x^4 P^n u_1(0) = (-1)^n \sum_{i \geq n} c_i \partial_x^4 f_{i-n}(0) + (-1)^n b_n \partial_x^4 g_0(0) + (-1)^n \sum_{i > n} b_i \partial_x^4 g_{i-n}(0) = (-1)^n b_n.$$

This leads us to define

$$c_n = (-1)^n \partial_x^3 P^n u_1(0) \quad \text{and} \quad b_n = (-1)^n \partial_x^4 P^n u_1(0) \quad \text{for all } n \geq 0. \quad (3-5)$$

**Claim.** There exist  $r \in (R_0, R) \cap (0, 4)$  and a constant  $K = K(r) > 0$  such that

$$|\partial_x^n u_1(x)| \leq K \frac{n!}{r^n} \quad \text{for all } x \in [-1, 0].$$

Since  $R > 2R_0$ , we can write  $R = 2R_0 + \alpha$  with  $\alpha > 0$ . On the other hand, by definition  $R_0 \approx 3.54$ ; for  $\kappa$  small enough, we have  $r := R_0 + \kappa < \min\{R_0 + \alpha, 4\}$ . Consequently,  $R_0 + r < 2R_0 + \alpha = R$  and  $\overline{D(w, r)} \subset \overline{D(0, R_0 + r)} \subset D(0, R)$  for all  $w \in \overline{D(0, R_0)}$ . Define  $K := \max\{|z(w)|; w \in \overline{D(0, R_0 + r)}\}$ , where  $z \in H(D(0, R))$  is such that  $z|_{[-1, 0]} = u_1$ . Under the hypothesis of  $u_1 \in \mathcal{R}$ , it follows that

$$|\partial_x^n u_1(x)| \leq K \frac{n!}{r^n} \quad \text{for all } x \in [-1, 0],$$

as desired, showing the claim.

Now, observe that for  $r < 4$ , the function

$$a_r : [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto a_r(x) = \frac{(x+3)!}{r^{x+3}}$$

is increasing in  $[0, \infty)$ . Thus, using this fact, Lemma 2.6 with  $p = \infty$ , the claim, and inequalities (2-3) and  $5n + 1 \leq 6n \leq 6 \cdot 2^n$ , we get

$$|c_n| \leq \sum_{i=0}^n \|P^i \partial_x^3 u_1\|_\infty \leq \frac{1}{2} 3^{n+1} \|\partial_x^3 u_1\|_{5n, \infty} \leq \frac{9 \cdot 2^3 \cdot 3!}{r^3} K \cdot \frac{6^n \cdot 2^{5n}}{r^{5n}} \cdot (5n)!,$$

and analogously,

$$|b_n| \leq \frac{9 \cdot 2^4 \cdot 4!}{r^4} K \cdot \frac{6^n \cdot 2^{5n}}{r^{5n}} \cdot (5n)!.$$

Therefore,

$$|c_n|, |b_n| \leq K' \left( \frac{6 \cdot 2^5}{r^5} \right)^n (5n)! \quad \text{for all } n \geq 0,$$

for some positive constant  $K' > 0$ .

Define  $H = 6 \cdot 2^5 / r^5$  and observe that  $r > R_0$  implies that  $He^{e^{-1}} < \frac{1}{2}$ . Choose  $\tilde{H} \in (He^{e^{-1}}, \frac{1}{2})$ . Then, from Proposition 3.2, there exist functions  $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{R})$  such that

$$\tilde{f}^{(n)}(0) = c_n, \quad \tilde{g}^{(n)}(0) = b_n, \quad \text{and} \quad |\tilde{f}^{(n)}(t)|, |\tilde{g}^{(n)}(t)| \leq K' \tilde{H}^n (5n)! \quad \text{for all } t \in \mathbb{R},$$

for every  $n \geq 0$ . Define  $f(t) = \tilde{f}(t - T)$  and  $g(t) = \tilde{g}(t - T)$ . Then  $f, g \in C^\infty(\mathbb{R})$  with, for all  $n \geq 0$ ,

$$f^{(n)}(t) = \tilde{f}^{(n)}(t - T) \quad \text{and} \quad g^{(n)}(t) = \tilde{g}^{(n)}(t - T) \quad \text{for all } t \in \mathbb{R}.$$

Moreover,  $f^{(n)}(T) = c_n$ ,  $g^{(n)}(T) = b_n$ , and

$$|f^{(n)}(t)|, |g^{(n)}(t)| \leq K' \tilde{H}^n (5n)! \quad \text{for all } n \geq 0 \text{ and all } t \in \mathbb{R}.$$

From the last inequality, we have

$$|f^{(n)}(t)|, |g^{(n)}(t)| \leq K' \frac{(5n)!}{R_3^{5n}} \quad \text{for all } n \geq 0 \text{ and all } t \in \mathbb{R}, \quad (3-6)$$

where  $R_3 = \tilde{H}^{-\frac{1}{5}} > 2^{\frac{1}{5}}$ . From (3-6), we have  $f, g \in G^5([0, T])$ . Indeed, from Stirling's formula,

$$(5n)! \sim 5^{5n+\frac{1}{2}} (\sqrt{2\pi n})^{-4} n!^5 = \frac{5^{\frac{1}{2}} (2\pi)^{-2}}{n^2} 5^{5n} n!^5.$$

Hence, for some positive constant  $K'' > 0$ , we have

$$|f^{(n)}(t)|, |g^{(n)}(t)| \leq K' \frac{(5n)!}{R_3^{5n}} \leq K'' \frac{5^{5n}}{R_3^{5n}} n!^5 = K'' \frac{n!^5}{(5^{-5} R_3^5)^n}. \quad (3-7)$$

Pick any  $\tau \in (0, T)$  and let

$$\beta(t) = 1 - \phi_s \left( \frac{t - \tau}{T - \tau} \right), \quad t \in [0, T].$$

Observe that

$$\beta^{(i)}(t) = -(T - \tau)^{-i} \phi_s^{(i)} \left( \frac{t - \tau}{T - \tau} \right), \quad i \geq 1.$$

By definition, we have  $\text{supp } \phi_s \subset (-\infty, 1)$ , so  $\beta(T) = 1$  and  $\beta^{(i)}(T) = 0$  for all  $i \geq 1$ . Define

$$y(t) = f(t)\beta(t) \quad \text{and} \quad z(t) = g(t)\beta(t), \quad t \in [0, T]. \quad (3-8)$$

From the Leibniz rule, we have

$$y^{(n)}(t) = \sum_{j=0}^n \binom{n}{j} f^{(j)}(t) \beta^{(n-j)}(t) \quad \text{and} \quad z^{(n)}(t) = \sum_{j=0}^n \binom{n}{j} g^{(j)}(t) \beta^{(n-j)}(t).$$

Then

$$y^{(n)}(T) = \sum_{j=0}^n \binom{n}{j} f^{(j)}(T) \beta^{(n-j)}(T) = f^{(n)}(T) \beta(T) = c_n \quad (3-9)$$

and

$$z^{(n)}(T) = \sum_{j=0}^n \binom{n}{j} g^{(j)}(T) \beta^{(n-j)}(T) = g^{(n)}(T) \beta(T) = b_n. \quad (3-10)$$

On the other hand, by definition  $\phi_s \equiv 1$  in  $(-\infty, 0]$ , so  $\beta \equiv 0$  in  $(-\infty, \tau)$ . Consequently,  $y^{(n)} \equiv z^{(n)} \equiv 0$  in  $(-\infty, \tau)$  for any  $n \geq 0$ . In particular,

$$y^{(n)}(0) = z^{(n)}(0) = 0 \quad \text{for all } n \geq 0. \quad (3-11)$$

Since  $\beta \in G^2([0, T])$  and  $f, g \in G^5([0, T])$ , we have  $y, z \in G^5([0, T])$ . From [Martin et al. 2016b, Lemma 3.7], the same constant  $R$  of  $f$  and  $g$  in the definition of  $G^5([0, T])$  works for  $y$  and  $z$ . Hence, from (3-7), there exists  $K''' > 0$  such that

$$|y^{(n)}(t)|, |z^{(n)}(t)| \leq K''' \frac{n!^5}{(5^{-5} R_3^5)^n} \quad \text{for all } n \geq 0 \text{ and all } t \in [0, T].$$

Using Stirling's formula we obtain  $(5n)! \sim 5^{5n+\frac{1}{2}} (\sqrt{2\pi n})^{-4} n!^5$ , and so

$$n!^5 \sim 5^{-(5n+\frac{1}{2})} (\sqrt{2\pi n})^4 (5n)! = \frac{(2\pi)^2}{5^{\frac{1}{2}}} n^2 \frac{(5n)!}{5^{5n}}.$$

Then, there exists a constant  $\bar{K} > 0$  such that

$$|y^{(n)}(t)|, |z^{(n)}(t)| \leq K''' \frac{1}{(5^{-5} R_3^5)^n} \bar{K} n^2 \frac{(5n)!}{5^{5n}} = K''' \bar{K} \cdot \frac{n^2 (5n)!}{R_3^{5n}}.$$

Since  $R_3 > 2^{\frac{1}{5}}$ , we can pick  $\rho \in (1, R_3 2^{-\frac{1}{5}})$  such that the sequence  $(n^2/\rho^{5n})_{n \in \mathbb{N}}$  is bounded. Indeed, using L'Hospital's rule, we see that

$$\lim_{x \rightarrow \infty} \frac{x^2}{\rho^{5x}} = \lim_{x \rightarrow \infty} \frac{2x}{(5 \ln \rho) \rho^{5x}} = \lim_{x \rightarrow \infty} \frac{2}{(5 \ln \rho)^2 \rho^{5x}} = 0.$$

Hence, there exists  $\bar{K}' > 0$  such that  $n^2 \leq \bar{K}' \rho^{5n}$ , and therefore

$$|y^{(n)}(t)|, |z^{(n)}(t)| \leq K''' \bar{K} \cdot \bar{K}' \cdot \frac{\rho^{5n} (5n)!}{R_3^{5n}} = K''' \bar{K} \cdot \bar{K}' \cdot \frac{(5n)!}{(R_3 \rho^{-1})^{5n}}.$$

Defining  $K'''' = K''' \bar{K} \cdot \bar{K}'$  and  $R'_3 = R_3/\rho$ , we have  $R'_3 > 2^{\frac{1}{5}}$  and

$$|y^{(n)}(t)|, |z^{(n)}(t)| \leq K'''' \frac{(5n)!}{(R'_3)^{5n}} \quad \text{for all } n \geq 0 \text{ and all } t \in [0, T]. \quad (3-12)$$

Let  $u$  be as in (2-4) corresponding to  $y$  and  $z$  given in (3-8). From (3-12) and by Proposition 3.1, we have that  $y \in G^{1,5}([-1, 0] \times [0, T])$ , and it solves (2-11). Furthermore, (3-11) gives us

$$u(x, 0) = \sum_{j \geq 0} f_j(x) y^{(j)}(0) + \sum_{j \geq 0} g_j(x) z^{(j)}(0) = 0.$$

Setting  $h_1 = u(-1, t)$  and  $h_2 = u_x(-1, t)$ , we get  $h_1, h_2 \in G^5([0, T])$ , and therefore  $u$  solves (1-1) with  $h_1$  and  $h_2$  as control inputs and  $y_0 = 0$  as initial data. From the proof of Proposition 3.1 we know that for all  $n, m \in \mathbb{N}$ , the series

$$\sum_{j \geq 0} \partial_t^m \partial_x^n (f_j(x) y^{(j)}(t) + g_j(x) z^{(j)}(t))$$

converges uniformly on  $[-1, 0] \times [0, T]$  to  $\partial_t^m \partial_x^n u$  and consequently, for all  $n, i \geq 0$ ,

$$\begin{aligned} P^n u(x, t) &= \sum_{j \geq 0} P^n f_j(x) y^{(j)}(t) + \sum_{j \geq 0} P^n g_j(x) z^{(j)}(t) \\ &= (-1)^n \sum_{j \geq n} f_{j-n}(x) y^{(j)}(t) + (-1)^n \sum_{j \geq n} g_{j-n}(x) z^{(j)}(t) \end{aligned}$$

and

$$\partial_x^i P^n u(x, t) = (-1)^n \sum_{j \geq n} \partial_x^i f_{j-n}(x) y^{(j)}(t) + (-1)^n \sum_{j \geq n} \partial_x^i g_{j-n}(x) z^{(j)}(t),$$

for every  $(x, t) \in [-1, 0] \times [0, T]$ . In particular, (3-9) and (3-10) give us that

$$\partial_x^i P^n u(x, T) = (-1)^n \sum_{j \geq n} c_j \partial_x^i f_{j-n}(x) + (-1)^n \sum_{j \geq n} b_j \partial_x^i g_{j-n}(x) \quad \text{for all } i, n \geq 0 \text{ and all } x \in [-1, 0].$$

Thus, (2-5)–(2-8) provide us with

$$\begin{aligned} P^n u(0, T) &= \partial_x P^n u(0, T) = \partial_x^2 P^n u(0, T) = 0, \\ \partial_x^3 P^n u(0, T) &= (-1)^n \sum_{j \geq n} c_j \partial_x^3 f_{j-n}(0) + (-1)^n \sum_{j \geq n} b_j \partial_x^3 g_{j-n}(0) = (-1)^n c_n, \\ \partial_x^4 P^n u(0, T) &= (-1)^n \sum_{j \geq n} c_j \partial_x^4 f_{j-n}(0) + (-1)^n \sum_{j \geq n} b_j \partial_x^4 g_{j-n}(0) = (-1)^n b_n, \end{aligned}$$

and

$$\begin{aligned} \partial_x^j P^n u(0, T) &= 0, \quad j = 0, 1, 2, \\ \partial_x^3 P^n u(0, T) &= \partial_x^3 P^n u_1(0), \quad \partial_x^4 P^n u(0, T) = \partial_x^4 P^n u_1(0). \end{aligned} \tag{3-13}$$

Define  $\psi \in G^1([-1, 0])$  by  $\psi(x) = u(x, T) - u_1(x)$  for all  $x \in [-1, 0]$ , and using the fact that  $u_1 \in \mathcal{R}_R$  together with (3-13), it holds that  $\partial_x^j P^n \psi(0) = 0$  for  $j = 0, 1, 2, 3, 4$ . From Lemma 3.3 it follows that  $\psi \equiv 0$ , that is,  $u(x, T) = u_1(x)$ , which concludes the proof.  $\square$

### Acknowledgments

The authors are grateful to the referee for the careful reading of this paper and their valuable suggestions and comments. This work is part of the Ph.D. thesis of da Silva at the Department of Mathematics of the Universidade Federal de Pernambuco.

### References

- [Alvarez-Samaniego and Lannes 2008] B. Alvarez-Samaniego and D. Lannes, “Large time existence for 3D water-waves and asymptotics”, *Invent. Math.* **171**:3 (2008), 485–541. [MR](#)
- [Araruna et al. 2012] F. D. Araruna, R. A. Capistrano-Filho, and G. G. Doronin, “Energy decay for the modified Kawahara equation posed in a bounded domain”, *J. Math. Anal. Appl.* **385**:2 (2012), 743–756. [MR](#)
- [Benney 1966] D. J. Benney, “Long waves on liquid films”, *J. Math. and Phys.* **45** (1966), 150–155. [MR](#)

- [Berloff and Howard 1997] N. G. Berloff and L. N. Howard, “Solitary and periodic solutions of nonlinear nonintegrable equations”, *Stud. Appl. Math.* **99**:1 (1997), 1–24. [MR](#)
- [Biswas 2009] A. Biswas, “Solitary wave solution for the generalized Kawahara equation”, *Appl. Math. Lett.* **22**:2 (2009), 208–210. [MR](#)
- [Bona et al. 2008] J. L. Bona, D. Lannes, and J.-C. Saut, “Asymptotic models for internal waves”, *J. Math. Pures Appl.* (9) **89**:6 (2008), 538–566. [MR](#)
- [Boyd 1991] J. P. Boyd, “Weakly non-local solitons for capillary-gravity waves: fifth-degree Korteweg–de Vries equation”, *Phys. D* **48**:1 (1991), 129–146.
- [Capistrano-Filho and Gomes 2021] R. A. Capistrano-Filho and M. M. S. Gomes, “Well-posedness and controllability of Kawahara equation in weighted Sobolev spaces”, *Nonlinear Anal.* **207** (2021), art. id. 112267. [MR](#)
- [Chen 2019] M. Chen, “Internal controllability of the Kawahara equation on a bounded domain”, *Nonlinear Anal.* **185** (2019), 356–373. [MR](#)
- [Chen and Rosier 2020] M. Chen and L. Rosier, “Exact controllability of the linear Zakharov–Kuznetsov equation”, *Discrete Contin. Dyn. Syst. Ser. B* **25**:10 (2020), 3889–3916. [MR](#)
- [Chen and Rosier 2022] M. Chen and L. Rosier, “Reachable states for the distributed control of the heat equation”, *C. R. Math. Acad. Sci. Paris* **360** (2022), 627–639. [MR](#)
- [Chen et al. 2022] S. Chen, Z. Du, J. Liu, and K. Wang, “The dynamic properties of a generalized Kawahara equation with Kuramoto–Sivashinsky perturbation”, *Discrete Contin. Dyn. Syst. Ser. B* **27**:3 (2022), 1471–1496. [MR](#)
- [Coclite and di Ruvo 2022] G. M. Coclite and L. di Ruvo, “On the solutions for a Benney–Lin type equation”, *Discrete Contin. Dyn. Syst. Ser. B* **27**:11 (2022), 6865–6883. [MR](#)
- [Cui et al. 2006] S. B. Cui, D. G. Deng, and S. P. Tao, “Global existence of solutions for the Cauchy problem of the Kawahara equation with  $L^2$  initial data”, *Acta Math. Sin. (Engl. Ser.)* **22**:5 (2006), 1457–1466. [MR](#)
- [Fliess and Join 2013] M. Fliess and C. Join, “Model-free control”, *Internat. J. Control* **86**:12 (2013), 2228–2252. [MR](#)
- [Fliess et al. 1995] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, “Flatness and defect of non-linear systems: introductory theory and examples”, *Internat. J. Control* **61**:6 (1995), 1327–1361. [MR](#)
- [Glass and Guerrero 2009] O. Glass and S. Guerrero, “On the controllability of the fifth-order Korteweg–de Vries equation”, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **26**:6 (2009), 2181–2209. [MR](#)
- [Hasimoto 1970] H. Hasimoto, “Water waves”, *Kagaku* **40**:8 (1970), 401–408. In Japanese.
- [Hunter and Scheurle 1988] J. K. Hunter and J. Scheurle, “Existence of perturbed solitary wave solutions to a model equation for water waves”, *Phys. D* **32**:2 (1988), 253–268. [MR](#)
- [Iguchi 2007] T. Iguchi, “A long wave approximation for capillary-gravity waves and the Kawahara equation”, *Bull. Inst. Math. Acad. Sin. (N.S.)* **2**:2 (2007), 179–220. [MR](#)
- [Jin 2009] L. Jin, “Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation”, *Math. Comput. Modelling* **49**:3-4 (2009), 573–578. [MR](#)
- [Kakutani 1960] T. Kakutani, “Axially symmetric stagnation-point flow of an electrically conducting fluid under transverse magnetic field”, *J. Phys. Soc. Japan* **15** (1960), 688–695. [MR](#)
- [Kawahara 1972] T. Kawahara, “Oscillatory solitary waves in dispersive media”, *J. Phys. Soc. Japan* **33**:1 (1972), 260–264.
- [Kaya and Al-Khaled 2007] D. Kaya and K. Al-Khaled, “A numerical comparison of a Kawahara equation”, *Phys. Lett. A* **363**:5-6 (2007), 433–439.
- [Lannes 2013] D. Lannes, *The water waves problem*, Mathematical Surveys and Monographs: Mathematical analysis and asymptotics **188**, Amer. Math. Soc., Providence, RI, 2013. [MR](#)
- [Laurent and Rosier 2020] C. Laurent and L. Rosier, “Exact controllability of semilinear heat equations in spaces of analytic functions”, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **37**:4 (2020), 1047–1073. [MR](#)
- [Lin 1974] S. P. Lin, “Finite amplitude side-band stability of a viscous film”, *J. Fluid Mechanics* **63**:3 (1974), 417–429.
- [Lions 1988] J.-L. Lions, “Exact controllability, stabilization and perturbations for distributed systems”, *SIAM Rev.* **30**:1 (1988), 1–68. [MR](#)

- [Martin et al. 2014] P. Martin, L. Rosier, and P. Rouchon, “Null controllability of the heat equation using flatness”, *Automatica J. IFAC* **50**:12 (2014), 3067–3076. [MR](#)
- [Martin et al. 2016a] P. Martin, L. Rosier, and P. Rouchon, “Null controllability of one-dimensional parabolic equations by the flatness approach”, *SIAM J. Control Optim.* **54**:1 (2016), 198–220. [MR](#)
- [Martin et al. 2016b] P. Martin, L. Rosier, and P. Rouchon, “On the reachable states for the boundary control of the heat equation”, *Appl. Math. Res. Express. AMRX* **2016**:2 (2016), 181–216. [MR](#)
- [Martin et al. 2018] P. Martin, L. Rosier, and P. Rouchon, “Controllability of the 1D Schrödinger equation using flatness”, *Automatica J. IFAC* **91** (2018), 208–216. [MR](#)
- [Martin et al. 2019] P. Martin, I. Rivas, L. Rosier, and P. Rouchon, “Exact controllability of a linear Korteweg–de Vries equation by the flatness approach”, *SIAM J. Control Optim.* **57**:4 (2019), 2467–2486. [MR](#)
- [Petit and Rouchon 2001] N. Petit and P. Rouchon, “Flatness of heavy chain systems”, *SIAM J. Control Optim.* **40**:2 (2001), 475–495. [MR](#)
- [Polat et al. 2006] N. Polat, D. Kaya, and H. I. Tutarlar, “A analytic and numerical solution to a modified Kawahara equation and a convergence analysis of the method”, *Appl. Math. Comput.* **179**:2 (2006), 466–472. [MR](#)
- [Pomeau et al. 1988] Y. Pomeau, A. Ramani, and B. Grammaticos, “Structural stability of the Korteweg–de Vries solitons under a singular perturbation”, *Phys. D* **31**:1 (1988), 127–134. [MR](#)
- [Vasconcellos and da Silva 2008] C. F. Vasconcellos and P. N. da Silva, “Stabilization of the linear Kawahara equation with localized damping”, *Asymptot. Anal.* **58**:4 (2008), 229–252. Correction in **66**:4 (2010), 119–124. [MR](#)
- [Yusufoğlu et al. 2008] E. Yusufoğlu, A. Bekir, and M. Alp, “Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using sine-cosine method”, *Chaos Solitons Fractals* **37**:4 (2008), 1193–1197. [MR](#)
- [Zhao et al. 2018] X. Zhao, X. Zhang, and H. Feng, “Global well-posedness and exponential decay for fifth-order Korteweg–de Vries equation posed on the finite domain”, *J. Math. Anal. Appl.* **468**:2 (2018), 976–997. [MR](#)
- [Zhou 2019] D. Zhou, “On the Cauchy problem for the strong dissipative fifth-order KdV equations”, *Nonlinear Anal. Real World Appl.* **48** (2019), 466–484. [MR](#)

Received 2 Apr 2025. Revised 26 Jun 2025.

ROBERTO DE A. CAPISTRANO-FILHO:

[roberto.capistranofilho@ufpe.br](mailto:roberto.capistranofilho@ufpe.br)

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Brazil

JANDEILSON SANTOS DA SILVA:

[jandeilson.santos@ufpe.br](mailto:jandeilson.santos@ufpe.br)

Departamento de Matemática, Universidade Federal de Pernambuco, Recife, Brazil

# Tunisian Journal of Mathematics

[msp.org/tunis](http://msp.org/tunis)

## EDITORS-IN-CHIEF

Ahmed Abbes CNRS & IHES, France  
[abbes@ihes.fr](mailto:abbes@ihes.fr)  
Ali Baklouti Faculté des Sciences de Sfax, Tunisia  
[ali.baklouti@fss.usf.tn](mailto:ali.baklouti@fss.usf.tn)

## EDITORIAL BOARD

Hajer Bahouri CNRS & LAMA, Université Paris-Est Créteil, France  
Arnaud Beauville Laboratoire J. A. Dieudonné, Université Côte d'Azur, France  
Philippe Biane CNRS & Université Paris-Est, France  
Christophe Breuil CNRS & Université Paris-Saclay, France  
Ewa Damek University of Wrocław, Poland  
Bassam Fayad CNRS & Inst. Math. Jussieu - Paris Rive Gauche, France  
Benoît Fresse Université Lille 1, France  
Dennis Gaiatsory Max Planck Institutes, Germany  
Bernhard Hanke Universität Augsburg, Germany  
Emmanuel Hebey Université de Cergy-Pontoise, France  
Michael A. Hill University of Minnesota, United States  
Mohamed Ali Jendoubi Université de Carthage, Tunisia  
Sadok Kallel Université de Lille 1, France & American Univ. of Sharjah, UAE  
Minhyong Kim ICMS, University of Edinburgh, UK & KIAS, Seoul, Korea  
Toshiyuki Kobayashi The University of Tokyo & Kavli IPMU, Japan  
Patrice Le Calvez Inst. Math. Jussieu - Paris Rive Gauche & Sorbonne Univ., France  
Jérôme Le Rousseau Université Sorbonne Paris Nord, France  
Amine Marrakchi CNRS & ENS de Lyon, UMPA, France  
Nader Masmoudi Courant Institute, New York University, United States  
Haynes R. Miller Massachusetts Institute of Technology, United States  
Philibert Nang École Normale Supérieure de Libreville, Gabon  
Marc Peigné Institut Poisson, CNRS & Université de Tours, France  
Enrique Pujals City University of New York, United States  
Jean-François Quint CNRS – IMB, Université de Bordeaux, France  
Yiannis Sakellaridis Johns Hopkins University, United States  
Daniel Tataru University of California, Berkeley, United States  
Michał Wrochna Cergy Paris Université, France  
Hatem Zaag CNRS & Université Sorbonne Paris Nord, France

**PRODUCTION** [production@msp.org](mailto:production@msp.org)

The Tunisian Journal of Mathematics is an international publication published in electronic and print formats by MSP in Berkeley, and organized by the Tunisian Mathematical Society (<http://www.tms.mu.tn>).

See inside back cover or [msp.org/tunis](http://msp.org/tunis) for submission instructions.

The subscription price for 2026 is US \$445/year for the electronic version, and \$530/year (+\$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004 is published continuously online.

TJM peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY  
 **mathematical sciences publishers**  
nonprofit scientific publishing  
<http://msp.org/>

© 2026 Mathematical Sciences Publishers

|                                                               |     |
|---------------------------------------------------------------|-----|
| Reduced symplectic homology and string topology               | 203 |
| KAI CIELIEBAK and ALEXANDRU OANCEA                            |     |
| Twisted differential KO-theory                                | 265 |
| DANIEL GRADY and HISHAM SATI                                  |     |
| An extension of Bochner's theorem                             | 305 |
| LAMINE TIMITÉ and IBRAHIMA TOURÉ                              |     |
| Bounding ramification with coherent sheaves                   | 321 |
| HAOYU HU and JEAN-BAPTISTE TEYSSIER                           |     |
| Control of Kawahara equation using flat outputs               | 351 |
| ROBERTO DE A. CAPISTRANO-FILHO and JANDEILSON SANTOS DA SILVA |     |